Moscow University Physics Bulletin
Vol. 50, No. 5, pp. 15–18, 1995

Vestnik Moskovskogo Universiteta. Fizika
UDC 530.12:550.831

"FREE" EXPANSION OF A POWDERED SCALAR BALL WITH AN ELECTRIC CHARGE

A. A. Vlasov

An exact solution for an expanding ball of particles with a scalar and an electric charge is found. This solution corresponds to ball expansion without acceleration. Special features of the solution are discussed.

The model of a scalar field is simple and therefore useful in detecting and examining some special features in the behavior of classical field schemes, including relativistic gravitation theory [1]. In particular, it was shown in [2] that for an expanding shell, the scalar and gravitational fields exhibit qualitatively similar behaviors. This work considers another problem, which may prove useful in discovering details of a spherically symmetric collapse in relativistic gravitation theory. This is the problem of the behavior of a ball of scalar particles whose mutual gravitation is in the scalar field model compensated to a degree by the presence of like particle charges.

Let n be the density of particles with the scalar charge Q and electric charge e. If u^p is the 4-velocity of the particles ($u^p = dx^i/ds = u^0(1, \mathbf{v})$, where ds is the interval in the Minkowski space), then according to the law of conservation of particle density,

$$D_m(nu^m) = 0, \tag{1}$$

where D is the derivative covariant with respect to the metric γ_{mn} of the Minkowski space.

The Lagrangian of the model of such particles, which interact with the electromagnetic field of their own, is

$$L = D_p\varphi D^p\varphi + nQ\varphi - nQ - A_p u^p ne - \frac{1}{16\pi}F_{mn}F^{mn}, \tag{2}$$

where φ is the scalar field, A^p is the 4-potential of the electromagnetic field, F_{mn} is the Maxwell tensor, and $j^p = neu^p$ is the 4-density of the electromagnetic current.

To Lagrangian (2) there correspond the field equations

$$\begin{aligned} D_pD^p\varphi &= 4\pi Qn, \\ D_mF^{mp} &= 4\pi j^p \end{aligned} \tag{3}$$

and the complete energy-momentum tensor

$$\begin{aligned} T^{mn} = Q(1-\varphi)nu^m u^n &+ \frac{1}{4\pi}\left[D^m\varphi D^n\varphi - \frac{1}{2}\gamma^{mn}D_p\varphi D^p\varphi\right] \\ &+ \frac{1}{4\pi}\left[-F^{mp}F^n_p + \frac{1}{4}\gamma^{mn}F^{pq}F_{pq}\right]. \end{aligned} \tag{4}$$

The tensor T^{mn} satisfies the differential conservation law

$$D_mT^{mn} = 0. \tag{5}$$

Equation (5) can be written as

$$Q(1-\varphi)\frac{d\mathbf{v}}{ds} = e(\mathbf{E} - \mathbf{v}(\mathbf{E}\mathbf{v}) + [\mathbf{v},\, \mathbf{H}]) + \frac{Q}{u^0}(\nabla + \mathbf{v}\partial_0)\varphi, \tag{6}$$

where **E** and **H** are the electromagnetic fields satisfying the standard Maxwell equations for the charge density enu^0 and the current density $enu^0\mathbf{v}$.

Applying the Gauss–Ostrogradsky theorem to the Maxwell equations for a spherically symmetric powdered ball that comprises particles with the charges Q and e gives

$$\mathbf{H} = 0, \quad \mathbf{E} = e(t,\, r)\frac{\mathbf{r}}{r^3}, \quad e(t,\, r) = 4\pi e\int_0^r drr^2nu^0. \tag{7}$$

For a nonstatic system, problem (3)–(7) is essentially nonlinear. However, there exists an exact partial solution to Eqs. (3)–(7). This solution is classified with so-called self-similar solutions:

$$\mathbf{v} = \mathbf{r}/t. \tag{8}$$

Expression (8) automatically makes the acceleration zero: $d\mathbf{v}/ds = 0$, i.e., we have free expansion of the ball according to law (8).

Setting $nu^0 = \sigma(t, r)/r^3$ and applying (8) to (1), we obtain

$$\sigma(t,\, r) = \sigma(r/t). \tag{9}$$

Dependence (9) allows us to write

$$e(t,\, r) = e(r/t) = 4\pi e\int_0^{r/t} dy\frac{\sigma(y)}{y}. \tag{10}$$

Introducing into (3) and (6) the notation $r/t = z$ and $Q\varphi = \chi/r + \mathrm{const}$ and using (8)–(10), we can represent (3)–(7) as

$$-\chi + \frac{d\chi}{dz}z(1-z^2) = -4\pi e^2\sqrt{1-z^2}\int_0^z dy\frac{\sigma(y)}{y},$$

$$\frac{d^2\chi}{dz^2}(z^2-1) + \frac{d\chi}{dz}2z = 4\pi Q^2\frac{\sigma(z)}{z}\sqrt{1-z^2}. \tag{11}$$

An exact solution to (11),which is bounded for $r = 0$, is

$$\varphi = \mathrm{const} + A\frac{4\pi(e^2-Q^2)}{Qr}\left[\left(\frac{1+z}{1-z}\right)^{\nu} - \left(\frac{1-z}{1+z}\right)^{\nu}\right],$$

$$\sigma = z^2(1-z^2)^{-3/2}A\left[\left(\frac{1+z}{1-z}\right)^{\nu} - \left(\frac{1-z}{1+z}\right)^{\nu}\right], \tag{12}$$

where $\nu = |Q|/(2\sqrt{Q^2-e^2})$ and A is a free parameter related to the total number of particles in the ball.

Solution (12) is inner. By virtue of (8), the equation of the motion of the ball boundary is $r = v_0t$. Equations (3) and the asymptotics $\varphi \to 0$ for $r \to \infty$ imply that the general form of an outer solution is

$$\varphi = \frac{a(t-r)}{r} + \frac{b(t+r)}{r}.$$

Smoothly sewing the inner solution for φ with the outer, one makes it possible to find the functions $a(t-r)$ and $b(t+r)$ and set the constant in (12) equal to zero.

Thus, the outer solution is

$$\varphi = \frac{a_1}{r} + \frac{a_2}{r}\ln\left|\frac{t+r}{t-r}\cdot\frac{1-v_0}{1+v_0}\right|,$$
$$a_1 = A\frac{4\pi(e^2-Q^2)}{Q}\left[\left(\frac{1+v_0}{1-v_0}\right)^\nu - \left(\frac{1-v_0}{1+v_0}\right)^\nu\right],$$
$$a_2 = A\nu\frac{4\pi(e^2-Q^2)}{Q}\left[\left(\frac{1+v_0}{1-v_0}\right)^\nu + \left(\frac{1-v_0}{1+v_0}\right)^\nu\right]. \tag{13}$$

It follows from (13) that for $\mathrm{Im}(\nu) = 0$ ($Q^2 > e^2$), the outer solution diverges on the light lines $t = \pm r$. But for $\mathrm{Re}(\nu) = 0$ ($e^2 > Q^2$), this divergence can be obviated.

Indeed, in the notation

$$\nu = ki, \quad A\frac{4\pi(e^2-Q^2)}{Q} = -ia, \quad \frac{1+v_0}{1-v_0} = \exp\tau_0, \quad \frac{1+z}{1-z} = \exp\tau, \tag{14}$$

the complete solution for a scalar field is written as

$$r < v_0 t,$$
$$\varphi = \frac{2a}{r}\sin k\tau,$$
$$r > v_0 t,$$
$$\varphi = \frac{2a}{r}\sin k\tau_0 + \frac{2ak}{r}\cos k\tau_0 \ln\left|\frac{t+r}{t-r}\cdot\frac{1-v_0}{1+v_0}\right|, \tag{15}$$

and if $\cos k\tau_0 = 0$, then the velocity of the expansion of the ball boundary is

$$\ln\frac{1+v_0}{1-v_0} = \pi\frac{\sqrt{e^2-Q^2}}{|Q|} \tag{16}$$

for the specified charges, and the external solution does not contain divergences any longer and is static:

$$\varphi = 2a/r.$$

Replace the a parameter by the total number N of particles in the ball ($N = 4\pi\int_0^{v_0 t} dr\, r^2 n u^0$). By virtue of (14)–(16), we will eventually obtain the following complete solution for a scalar field:

$$\varphi_{\text{int}} = \frac{Q^*}{r}\sin\left(k\ln\left|\frac{t+r}{t-r}\right|\right),$$
$$n = \frac{Q^*Q}{4\pi(e^2-Q^2)}\frac{\sin\left(k\ln|(t+r)/(t-r)|\right)}{r(t^2-r^2)},$$
$$\varphi_{\text{ext}} = \frac{Q^*}{r}, \tag{17}$$

where

$$Q^* = NQ\frac{(\tau_0/\pi)^2+1}{\cosh(\tau_0/2)}, \quad \tau_0 = \ln\frac{1+v_0}{1-v_0}, \quad k = \frac{|Q|}{2\sqrt{e^2-Q^2}}.$$

For a scalar field, solution (17) is bounded in the entire space (the singularities of the interior solutions on the light lines lie outside the domain of the solution), and although (17) is written for $t > 0$, it can be extended to $t < 0$ (except for the singular point $t = r = 0$, which corresponds to contracting the ball to a point).

Note again that for the specified charges, the velocity of the expansion of the ball boundary is determined from (16) rather than is arbitrary.

Interestingly, solution (17) is "asymmetric": for $e^2 > Q^2$, the solution exists in form (8), but there is no solution for $e^2 < Q^2$ (in particular, for $e = 0$)!

The external solution for a scalar field is static, but the difference between the Q^* parameter and the total scalar charge NQ allows an outer observer to recognize the fact of expanding. Here again (as in relativistic gravitation theory), we see that, in the general case, spherically symmetric static solutions in classical field models are physically essentially different from spherically symmetric nonstatic solutions.

REFERENCES

1. A. A. Logunov and M. A. Mestvirishvili, *Relativistic Gravitation Theory* (in Russian), Moscow, 1989; A. A. Vlasov, *Noncalibration Approach in Relativistic Gravitation Theory* (in Russian), Moscow, 1992.
2. A. A. Vlasov, in: *Proc. 17th Workshop on Problems in High-Energy Physics and Field Theory*, Protvino, 1994.

30 January 1995

Department of Quantum Theory
and High-Energy Physics