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*Vestnik Moskovskogo*
*Universiteta. Fizika*
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# THEORETICAL AND MATHEMATICAL PHYSICS

# FOUNDATIONS OF POSSIBILITY THEORY. METHODS OF OPTIMAL ESTIMATION AND DECISION MAKING

# 1. MEASURE OF POSSIBILITY: DEFINITION AND PROPERTIES

Yu. P. Pyt'ev

**The paper presents a series of publications on the foundations of the possibility theory and its applications to mathematical modeling of reality on the basis of experimental data, knowledge and hypotheses of the researcher. In this work, the measure of possibilisty has been constructed and its properties have been considered.**

## INTRODUCTION

The probability-theoretic methods are extensively and successfully used in scientific investigations for modeling (in terms of randomness) various aspects of obscurity and indeterminacy reflecting the incompleteness and unreliability of knowledge and also for modeling fuzziness and inaccuracy characterizing the content of knowledge. While fuzziness and inaccuracy are naturally associated with probability distribution, obscurity and indeterminacy are related to partial lack of information about this distribution. This raises problems that are formulated in terms of the theory of testing statistical hypotheses [1] and estimation theory [2].

The probability-theoretic methods, however, appear to be ineffective in modeling a wide class of processes and phenomena whose organization is in the end determined by fuzziness and indeterminacy, e. g., complex physical, social, and economic systems, subjective inferences, etc. This is the reason why in the 1960–1970s, nonprobabilistic models of fuzziness and indeterminacy were the focus of considerable interest of scientists. Here is an incomplete list of the most important steps in the development of nonprobabilistic methods for modeling fuzziness and indeterminacy. They include the introduction by Savage [3] of subjective probability as a measure of uncertainty of an individual whose inferences satisfy certain "rationality" conditions, the upper and the lower Dempster probability [4] characterizing the incompleteness of observation and describing the indeterminacy in the probability theory simulated by many-valued maps and closely related to Choquet's capacity [5] and Shafer's likelihood and belief in the theory of decision [6] developed as a generalization of Dempster's ideas, and, lastly, the Zadeh possibility [7] based on the theory of fuzzy sets [8]. We should also mention Shackle's possibility [9] in his theory of decision and possibility and likelihood defined in terms of indeterminate fuzzy sets by the author [10].

The possibility theory is in the first place a natural generalization of the theory of errors* allowing gradations of possibilities of certain error values. On the other hand, the possibility theory, which makes it possible to formally characterize gradations of modalities of the "possible" and "necessary" types, can

* In the theory of errors, a measurement result is represented by a set of possible values of the measured characteristic.

also naturally be treated as a model of subjective inferences, in which more or less possible and necessary (reliable) events are represented together with other attributes of subjective inferences such as "some", "almost all", "approximately", "fairly accurately", "slightly", etc. These attributes are characteristic of the language of science.

The work gives a sketch of a possibility theory different from that described in [7, 11]. We also consider the applications of possibility theory to estimation and decision making problems. The construction exactly follows that of the probability theory, which makes it possible to trace formal analogies of the methods of the probability and possibility theories.

The linear countably additive* measure $p(\cdot)$ defined on some class $\mathcal{L}(X)$ of functions and taking on values in a given semiring $\mathcal{R}$ forms the basis. The possibility $P(\cdot)$ is defined by the $p(\cdot)$ values on the class $\{\chi_A(\cdot)\} \subset \mathcal{L}(X)$ of the characteristic functions of measurable subsets $A \subset X$ (events). The $p(f(\cdot))$ values on the other $f(\cdot) \in \mathcal{L}(X)$ functions determine the possibilities of the so-called fuzzy events, for which $f(\cdot) \in \mathcal{L}(X)$ are characteristic functions. By analogy with the standard probability theory, the notions of the possibilistic space, the integral over possibility, independence, conditional possibility, etc., are introduced.

The normative character of the probability theory, however, appears to be unwarranted for the possibility theory, because as distinguished from probability, the countably additive possibility is not continuous and can be extended with retention of all its properties from the $\sigma$-algebra of measurable subsets to the $\mathcal{R}(X)$ algebra of all subsets $X$, and the measure $p(\cdot)$ can be extended to the class of all functions defined on $X$ and taking on values in the semiring $\mathcal{R}$. In this work, we construct the only continuation, called maximum, of possibility to the $\mathcal{R}(X)$ algebra, which, in particular, allows any possibility to be characterized in terms of its distribution.

Next, we consider the methods of optimal estimation and decision based on minimizing the possibility (and/or necessity) of errors and losses, respectively.

The possibility theory methods described here as an alternative to probabilistic ones differ substantially from the latter. In the first place, the possibility of an event, as distinguished from its probability estimating the frequency of its occurrence in a regular stochastic experiment, is rather a relative estimate of its truth (or preferableness) in comparison with any other event; these estimates are obtained on a rank (gradation) scale, in which only the "greater", "less", or "equal" relations can be represented. We can say that the possibility theory *per se* and the possibilistic methods of optimal estimation and decision are invariant with respect to an arbitrary transformation of the scale of possibility values which leaves their order unchanged.

It follows that, generally speaking, possibility does not have direct interpretation in terms of events and their frequencies of occurrence inherent in probability and relating probability to experiment. The possibility theory, nevertheless, makes it possible to mathematically model the reality based on experimental observations and knowledge, hypotheses, and inferences of researchers, to check the correctness of constructed models, and to use them to estimate most efficiently the characteristics of processes and phenomena.

## 1. COUNTABLY ADDITIVE MEASURE WITH VALUES IN SEMIRING

Let $\mathcal{R}_{(p)}$ be the semiring $[0,1]$ with the addition operation "$+$" understood as "max" and the multiplication operation "$\cdot$" understood as "min",

$$a + b = \max(a,b), \quad a \cdot b = \min(a,b), \quad a, b \in [0,1].$$

The operations defined in such a way are commutative, $a + b = b + a$ and $a \cdot b = b \cdot a$, associative, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and mutually distributive,

$$a \cdot (b + c) = \min(a, \max(b,c)) = \max(\min(a,b), \min(a,c)) = (a \cdot b) + (a \cdot c); \tag{1}$$

$$a + (b \cdot c) = \max(a, \min(b,c)) = \min(\max(a,b), \max(a,c)) = (a + b) \cdot (a + c). \tag{2}$$

Let us define the neutral elements $\mathbf{0}$ and $\mathbf{1}$ of the semiring $\mathcal{R}_{(p)}$ as $\mathbf{0} = 0$ and $\mathbf{1} = 1$. We have

$$\begin{aligned} \mathbf{0} \cdot a = \min(0,a) = 0 = \mathbf{0}, \qquad \mathbf{0} + a = \max(0,a) = a, \\ \mathbf{1} \cdot a = \min(1,a) = a, \qquad \mathbf{1} + a = \max(a,1) = 1, \qquad a \in [0,1]. \end{aligned} \tag{3}$$

* With respect to addition understood as "max" and multiplication understood as "min".

The natural order on $\mathcal{R}_{(p)}$ defined by the inequality $\leq$ is consistent with the addition and multiplication operations

$$a \leq b \Rightarrow \begin{cases} a \cdot c \leq b \cdot c, \\ a \dotplus c \leq b \dotplus c, \end{cases} \qquad a, b, c \in \mathcal{R}_{(p)}; \quad \mathbf{0} < \mathbf{1}. \tag{4}$$

The sequence $\{a_n\} \subset \mathcal{R}_{(p)}$ will be called convergent if $\liminf_{n\to\infty} a_n = \sup_N \inf_{n\geq N} a_n = \inf_N \sup_{n\geq N} {}_{n\to\infty} a_n = a$; the number $a$ will be called the limit of this sequence, $a = \lim a_n$.

Let $\mathcal{L}(X)$ denote the class of functions defined on $X$ with values in $\mathcal{R}_{(p)}$, which

(1) together with each pair of functions $f(\cdot)$ and $g(\cdot)$ contains their sum $(f \dotplus g)(\cdot)$ and product $(f \cdot g)(\cdot)$,

$$\begin{aligned} (f \dotplus g)(x) &= f(x) \dotplus g(x) = \max(f(x), g(x)), \\ (f \cdot g)(x) &= f(x) \cdot g(x) = \min(f(x), g(x)), \end{aligned} \qquad x \in X; \tag{5}$$

(2) together with each function $f(\cdot)$ contains its "negation", $\neg f(x) = 1 \dot{-} f(x)$, $x \in X$;

(3) together with an arbitrary sequence of functions $f_1(\cdot)$, $f_2(\cdot), \ldots$ contains

$$\overset{\infty}{\underset{n=1}{\dotplus}} f_n(x) = \sup_n f_n(x), \quad \overset{\infty}{\underset{n=1}{\cdot}} f_n(x) = \inf_n f_n(x), \qquad x \in X, \tag{6}$$

and, therefore, its upper and lower limits, $\limsup_{n\to\infty} f_n(x) = \inf_N \sup_{n\geq N} f_n(x)$ and $\liminf_{n\to\infty} f_n(x) = \sup_N \inf_{n\geq N} f_n(x)$, $x \in X$, and the limit $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$, if the latter exists.

**Note 1.** In conditions 1 and 3, the first equalities are sufficient, because for arbitrary $f(\cdot)$ and $g(\cdot)$ functions from $\mathcal{L}(X) \neg(\neg f \dotplus \neg g)(\cdot) = (f \cdot g)(\cdot) \in \mathcal{L}(X)$, and the inclusion $\{f_n(\cdot)\} \subset \mathcal{L}(X)$ implies $\neg(\overset{\infty}{\underset{n=1}{\dotplus}} \neg f_n(\cdot)) = \overset{\infty}{\underset{n=1}{\cdot}} f_n(\cdot) \in \mathcal{L}(X)$.

**Definition 1.** We define measure $p(\cdot)$ as a countably additive function on $\mathcal{L}(X)$ taking on values in $\mathcal{R}_{(p)}$, i.e., such that $\forall a, b \in \mathcal{R}_{(p)}$ and $\forall f(\cdot), g(\cdot) \in \mathcal{L}(X)$,

$$p((a \cdot f(\cdot)) \dotplus (b \cdot g(\cdot))) = (a \cdot p(f(\cdot))) \dotplus (b \cdot p(g(\cdot))) \tag{7}$$

and $\forall \{f_n(\cdot)\} \subset \mathcal{L}(X)$*

$$p(\sup_n f_n(\cdot)) = p(\overset{\infty}{\underset{n=1}{\dotplus}} f_n(\cdot)) = \overset{\infty}{\underset{n=1}{\dotplus}} p(f_n(\cdot)) = \sup_n p(f_n(\cdot)). \tag{8}$$

Put $h(x) = \max(f(x), g(x))$, $x \in X$. According to condition (7), $p(h(\cdot)) = \max(p(f(\cdot)), p(g(\cdot)))$. It follows that if $f(x) \geq g(x)$, $x \in X$, then

$$p(h(\cdot)) = p(f(\cdot)) = \max(p(f(\cdot)), p(g(\cdot))) \geq p(g(\cdot)). \tag{9}$$

We can say that $p(\cdot)$ is a monotone nondecreasing function.

Let $\{f_n(\cdot)\} \subset \mathcal{L}(X)$ be a monotone nondecreasing sequence, i.e., $f_n(x) \leq f_{n+1}(x)$, $n = 1, 2, \ldots$, $x \in X$. It converges pointwise, and its limit $f(x) = \lim_{n\to\infty} f_n(x) = \sup_n f_n(x)$, $x \in X$, is contained in $\mathcal{L}(X)$. Conditions (8) and (9) fix continuity of $p(\cdot)$ with respect to such convergence, *viz.*, $p(f(\cdot)) = p(\lim_{n\to\infty} f_n(\cdot)) = p(\sup_n f_n(\cdot)) = \sup_n (p(f_n(\cdot)) = \lim_{n\to\infty} (p(f_n(\cdot))$. The last equality follows from $p(\cdot)$ being a monotone function, see (9).

As according to (9), $p(f_k(\cdot)) \geq p(\inf_{n\geq N} f_n(\cdot))$, $N = 1, 2, \ldots$, for any $k \geq N$, we have

$$\inf_{k\geq N} p(f_k(\cdot)) \geq p(\inf_{n\geq N} f_n(\cdot)), \quad N = 1, 2, \ldots, \tag{10}$$

* Measure $p(\cdot)$ is an analog of the Radon measure on $X$.

and, therefore, by virtue of (8) and (10),

$$\sup_N p(\inf_{n\geq N} f_n(\cdot)) = p(\sup_N \inf_{n\geq N} f_n(\cdot)) \equiv p(\liminf_{n\to\infty} f_n(\cdot)) \leq \sup_N \inf_{n\geq N} p(f_n(\cdot)) = \liminf_{n\to\infty} p(f_n(\cdot)).$$

In particular, for an arbitrary convergent sequence $f_n(\cdot) \subset \mathcal{L}(X)$, $n = 1, 2, \ldots$,

$$p(\lim_{n\to\infty} f_n(\cdot)) \leq \liminf_{n\to\infty} p(f_n(\cdot)), \tag{11}$$

i.e., generally, $p(\cdot)$ is only lower semicontinuous (with respect to pointwise convergence).

**Theorem 1.** *Measure $p(\cdot)$ has the following properties. This measure is*

(1) *monotone nondecreasing on $\mathcal{L}(X)$: if $f(\cdot) \geq g(\cdot)$ means that $f(x) \geq g(x)$, $x \in X$, then $f(\cdot) \geq g(\cdot) \Rightarrow p(f(\cdot)) \geq p(g(\cdot))$,*

(2) *continuous with respect to convergence of a monotone nondecreasing sequence: $f_{n+1}(\cdot) \geq f_n(\cdot)$, $n = 1, 2, \ldots \Rightarrow p(\lim_{n\to\infty} f_n(\cdot)) \leq \lim_{n\to\infty} p(f_n(\cdot))$,*

(3) *lower semicontinuous: $\{f_n(\cdot)\} \subset \mathcal{L}(X) \Rightarrow p(\liminf_{n\to\infty} f_n(\cdot)) \leq \liminf_{n\to\infty} p(f_n(\cdot))$, and if, in particular, $\{f_n(\cdot)\}$ converges, then $p(\lim_{n\to\infty} f_n(\cdot)) \leq \lim_{n\to\infty} p(f_n(\cdot))$.** ■

**Example 1.** Using sup as the integral and min as the product, we define measure $p(f(\cdot)) = p_\varphi(f(\cdot))$ as the scalar product of a fixed function $\varphi(\cdot) \in \mathcal{L}(X)$ on $f(\cdot) \in \mathcal{L}(X)$

$$p_\varphi(f(\cdot)) = \sup_{x\in X} \min(f(x), \varphi(x)), \qquad f(\cdot) \in \mathcal{L}(X). \tag{12}$$

Here, the $p_\varphi(\cdot)$ function is linear, because $\forall a, b \in \mathcal{R}_{(p)}$, $f(\cdot), g(\cdot) \in \mathcal{L}(X)$,

$$\begin{aligned} p_\varphi((a \cdot f(\cdot)) \dotplus ((b \cdot g(\cdot))) &= \sup_{x\in X} \min\{\max[\min(a, f(x)), \min(b, g(x))], \varphi(x)\} \\ &= \max\{\min(a, \sup_{x\in X} \min(f(x), \varphi(x))), \min(b, \sup_{x\in X} \min(g(x), \varphi(x)))\} \\ &= (a \cdot p_\varphi(f(\cdot))) \dotplus (b \cdot p_\varphi(g(\cdot))), \end{aligned}$$

and countably additive, because $p_\varphi(\overset{\infty}{\underset{j=1}{\dotplus}} f_j(x)) = \sup_{x\in X} \min(\sup_j f_j(x), \varphi(x)) = \sup_j \sup_{x\in X} \min(f_j(x), \varphi(x)) = \sup_j p_\varphi(f_j(\cdot)) = \overset{\infty}{\underset{j=1}{\dotplus}} p_\varphi(f_j(\cdot))$.

At the same time, $p_\varphi(\lim_{n\to\infty} f_n(\cdot)) = \sup_{x\in X} \min(\sup_N \inf_{n\geq N} f_n(x), \varphi(x) \leq \sup_N \inf_{n\geq N} \sup_{x\in X} \min(f_n(x), \varphi(x)) = \liminf_{n\to\infty} p_\varphi(f_n(\cdot))$, i.e., like the general measure $p(\cdot)$, the $p_\varphi(\cdot))$ one is only lower semicontinuous. ■

It will, however, be shown that equality (12) is the general expression for the properly continued measure $p(\cdot)$.

Note that $p_{\overset{\infty}{\underset{j=1}{\dotplus}} \varphi_j}(f(\cdot)) = \overset{\infty}{\underset{j=1}{\dotplus}} p_{\varphi_j}(f(\cdot))$, $p_{\overset{\infty}{\underset{j=1}{\dot\cdot}} \varphi_j}(f(\cdot)) \leq \overset{\infty}{\underset{j=1}{\dot\cdot}} p_{\varphi_j}(f(\cdot))$.

## 2. MEASURE OF POSSIBILITY: DEFINITION AND PROPERTIES

Let $\mathcal{A}$ be some class of subsets of $X$, $\mathcal{L}(X)$ be a minimum (by inclusion) class of functions containing all step functions, i.e., functions of the form

$$\begin{gathered} f(x) = \overset{n}{\underset{k=1}{\dotplus}} (c_k \cdot \chi_{A_k}(x)) = \max_{1\leq k\leq n} \min(c_k, \chi_{A_k}(x)), \\ x \in X, \quad A_k \in \mathcal{A}, \quad k = 1, \ldots, n, \quad n = 1, 2, \ldots, \end{gathered} \tag{13}$$

where $c_k \in [0, 1]$, $\chi_{A_k}(\cdot)$ is the characteristic function of the set $A_k \subset X$: $\chi_{A_k}(x) = 1$, if $x \in A_k$, $\chi_{A_k}(x) = 0$, if $x \notin A_k$, $A_k \cap A_p = \varnothing$, if $k \neq p$, $k, p = 1, 2, \ldots, n$, $X = \bigcup_{k=1}^{n} A_k$, $n = 1, 2, \ldots$. As for the class

* The analogs of properties 2 and 3 in integration theory are the Lebesgue monotone convergence theorem and the Fatou lemma, respectively.

$\mathcal{A}$ of sets $A_1, \ldots, A_n$, $n = 1, 2, \ldots$, present in (13), it should be closed with respect to all set-theoretic operations. Indeed, because $\mathcal{L}(X)$ contains characteristic functions, for instance, $\chi_A(\cdot)$, $\chi_B(\cdot)$, it also contains $\chi_{A\cup B}(\cdot) = \max(\chi_A(\cdot), \chi_B(\cdot))$, $\chi_{A\cap B}(\cdot) = \min(\chi_A(\cdot), \chi_B(\cdot))$, and $\neg\chi_A(\cdot) = \chi_{X\setminus A}(\cdot)$; it follows that if $A, B \in \mathcal{A}$, then also $A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$, and $X \setminus A \in \mathcal{A}$. If $\{\chi_{A_n}(\cdot)\} \subset \mathcal{L}(X)$, then $\chi_{\bigcup_n A_n}(\cdot) = \sup_n \chi_{A_n}(\cdot) \in \mathcal{L}(X)$, $\chi_{\bigcap_n A_n}(\cdot) = \inf_n \chi_{A_n}(\cdot) \in \mathcal{L}(X)$, and if the sequence $\chi_{A_1}(\cdot), \chi_{A_2}(\cdot), \ldots$ converges, then $\lim\limits_{n\to\infty} \chi_{A_n}(\cdot) = \chi_{\lim\limits_{n\to\infty} A_n}(\cdot) \in \mathcal{L}(X)$, where $\lim\limits_{n\to\infty} A_n = \bigcup\limits_N \bigcap\limits_{n\geq N} A_n = \bigcap\limits_N \bigcup\limits_{n\geq N} A_n$ also belongs to the class of sets $\mathcal{A}$, which, as follows from its properties specified above, is a $\sigma$-algebra.

Let $\{f_n(\cdot)\}$ be a sequence of piecewise constant and, therefore, $\mathcal{A}$-measurable functions. Then $\sup\limits_n f_n(\cdot)$, $\inf\limits_n f_n(\cdot)$, $\liminf\limits_{n\to\infty} f_n(\cdot)$, and $\limsup\limits_{n\to\infty} f_n(\cdot)$ are $\mathcal{A}$-measurable. If $\{f_n(\cdot)\}$ is a sequence of $\mathcal{A}$-measurable functions, then these operations yield $\mathcal{A}$-measurable functions again. To show that $\mathcal{L}(X)$ is a class of $\mathcal{A}$-measurable functions defined on $X$ and taking values in $[0, 1]$, it remains to note that each such $f(\cdot)$ function can be represented as the limit of a uniformly convergent monotone sequence of piecewise constant $\mathcal{A}$-measurable functions. Namely, the sequences

$$\begin{aligned} \underline{f}_n(x) &= \mathop{\dot{+}}_{k=1}^{n} \left(\alpha_k^{(n)} \cdot \chi_{A_k}^{(n)}(x)\right), \quad n = 1, 2, \ldots, \quad x \in X, \\ \bar{f}_n(x) &= \mathop{\dot{+}}_{k=1}^{n} \left(\alpha_{k-1}^{(n)} \cdot \chi_{A_k}^{(n)}(x)\right), \quad n = 1, 2, \ldots, \quad x \in X, \end{aligned} \tag{14}$$

uniformly converge to $f(x)$, $x \in X$, and the first one is monotone nondecreasing, whereas the second sequence is monotone nonincreasing, if $A_k^{(n)} = \{x \in X, \alpha_k^{(n)} \leq f(x) < \alpha_{k-1}^{(n)}\}$, $k = 1, 2, \ldots, n$, $1 = \alpha_0^{(n)} \geq \alpha_1^{(n)} \geq \ldots \geq \alpha_n^{(n)} = 0$, and $\varepsilon^{(n)} = \max\limits_{1\leq k\leq n}(\alpha_{k-1}^{(n)} - \alpha_k^{(n)}) \to 0$, when $n \to \infty$. Under these conditions, $\sup\limits_{x\in X}(f(x) - \underline{f}_n(x)) \leq \varepsilon^{(n)}$ and $\sup\limits_{x\in X}(\bar{f}_n(x) - f(x)) \leq \varepsilon^{(n)}$.

Any set $A \in \mathcal{A}$ is called $\mathcal{A}$-measurable or an event; an event can be specified by its characteristic function $\chi_A(\cdot) \in \mathcal{L}(X)$. Any $f(\cdot) \in \mathcal{L}(X)$ function determines a fuzzy event (fuzzy set [8]) and is called its characteristic function.*

**Definition 2.** The $P(A) = p(\chi_A(\cdot))$ value will be called the measure of possibility of the event $A \in \mathcal{A}$, or, shortly, the possibility of $A$. Accordingly, the $p(f(\cdot))$ value will be called the possibility of the fuzzy event specified by the characteristic function $f(\cdot) \in \mathcal{L}(X)$.

**Theorem 2.** *The possibility $P(A)$, $A \in \mathcal{A}$, has the following properties.*

1.

$$P(A \cup B) = p((\chi_A \dot{+} \chi_B)(\cdot)) = \max(P(A), P(B)), \qquad A, B \in \mathcal{A} \tag{15}$$

*and, as a consequence, $P(X) = P((X \setminus A) \cup A) = \max(P(X \setminus A), P(A))$, $A \in \mathcal{A}$. In addition, $P(A) \leq P(B)$, if $A \subset B$ (monotone property of possibility) and, as a consequence, $P(A \cap B) = p((\chi_A \cdot \chi_B)(\cdot)) \leq \min(P(A), P(B))$, $A, B \in \mathcal{A}$.*

2. *For any sequence of events $A_1, A_2, \ldots$*

$$P(\bigcup_{n=1}^{\infty} A_n) = \sup_n P(A_n) = \mathop{\dot{+}}_{n=1}^{\infty} P(A_n), \tag{16}$$

*i.e., the possibility $P(\cdot)$ is countably additive. If $A_1 \subset A_2 \subset \ldots$ and $A = \bigcup\limits_{n=1}^{\infty} A_n = \lim\limits_{n\to\infty} A_n$, then, as a consequence of the countably additive and monotone properties, $P(\cdot)$ is continuous with respect to this convergence, i.e., $P(A) = \sup\limits_n P(A_n) = \lim\limits_{n\to\infty} P(A_n)$.*

3. *Generally, if $A = \lim\limits_{n\to\infty} A_n = \bigcup\limits_N \bigcap\limits_{n\geq N} A_n = \bigcap\limits_N \bigcup\limits_{n\geq N} A_n$, then*

$$P(A) = \sup_N P(\bigcap_{n\geq N} A_n) \leq \sup_N \inf_{n\geq N} P(A_n) \equiv \liminf_{n\to\infty} P(A_n), \tag{17}$$

* As $f(\cdot) \in \mathcal{L}(X)$ is a characteristic function of a fuzzy subset of $X$, $\mathcal{L}(X)$ can be called fuzzy $\sigma$-algebra, $(N, \mathcal{L}(X))$, a fuzzy measurable space, and the elements of $\mathcal{L}(X)$, fuzzy measurable sets [12].

*i.e., $P(\cdot)$ is lower semicontinuous. In particular, if $A_1 \supset A_2 \supset \ldots$, then $A = \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$, and* $P(\bigcap_{n=1}^{\infty} A_n) \leq \lim_{n\to\infty} P(A_n)$. ■

Note that from the point of view of their possibility, any two events $A, B \in \mathcal{A}$ are similar to incompatible events of the probability theory, because $P(A \cup B) = \max(P(A), P(B)) = P(A) + P(B)$.

The third property of possibility means that the $P(\varnothing)$ value cannot be defined by continuity, because $P(A)$ is not continuous at $A = \varnothing$; if $\varnothing = \lim_{n\to\infty}(A_n)$, then $P(\varnothing) \leq \liminf_{n\to\infty} P(A_n)$, and if $A_1 \supset A_2 \supset \ldots$ and $A_n \downarrow \varnothing$, $n \to \infty$, then we obtain $P(\varnothing) \leq \lim_{n\to\infty} P(A_n)$. The $P(\varnothing)$ value can be defined as an arbitrary number from $[0, \inf_{x\in\mathcal{A}} P(A)]$. Then for any event $A$, $P(A \cup \varnothing) = \max(P(A), P(\varnothing)) = P(A)$ and $P(A \cap \varnothing) = \min(P(A), P(\varnothing)) = P(\varnothing)$. Further, we assume that $P(\varnothing) = 0$, unless otherwise stated.

On the other hand, if $X = \lim_{n\to\infty} A_n$, then according to (17), $P(X) \leq \liminf_{n\to\infty} P(A_n)$. At the same time, $P(A_n) \leq P(X)$, $n = 1, 2, \ldots$, and therefore $P(X) \geq \limsup_{n\to\infty} P(A_n)$. It follows that for any sequence $A_1, A_2, \ldots$ converging to $X$, $P(X) = \liminf_{n\to\infty} P(A_n) = \limsup_{n\to\infty} P(A_n)$, i.e., the sequence $P(A_1), P(A_2), \ldots$ converges to $P(X)$, and the possibility $P(\cdot)$ is continuous in $X$. It is natural to put $P(X) = 1$.

Further, any $P(A)$, $A \in \mathcal{A}$, function possessing the properties specified in Theorem 2 will be called possibility.

*The triple $(X, \mathcal{A}, P(\cdot))$ will be called possibilistic space.*

**Example 2.** Let us return to the measure $p(\cdot) = p_\varphi(\cdot)$ (12) and note that in this case, the possibility of the event $A \in \mathcal{A}$, $A \neq \varnothing$ is given by the equality

$$P(A) = P_\varphi(A) = \sup_{x\in A} \varphi(x), \quad P(\varnothing) = 0.$$

Here, $\varphi(\cdot) \in \mathcal{L}(X)$, and $\sup_{x\in X} \varphi(x) = 1$. This function can naturally be called possibility $P(\cdot)$ distribution. It will be shown that the possibility $p(\cdot)$ can always be continued to the $\mathcal{P}(X)$ algebra of all subsets of $X$ and specified in the form of its distribution. ■

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Department of Computer Methods in Physics