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APPROXIMATE CALCULATION OF GREEN'S FUNCTION FOR THE POISSON EQUATION ON SPACE $E_{D-2} \times V_2$

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The feasibility of approximately calculating the Green function for the Poisson equation on the space $E_{D-2} \times V_2$ and its regularized value in the limit of coinciding points is studied.

In field theory, calculations of local observables involve calculations of the Green function and its derivatives in the limit of coinciding points. Even for the Minkowski space, the corresponding values are known to diverge, which requires the use of some regularization procedure. The situation grows even more complex against a curved space-time background, because nonzero local curvature causes the appearance of additional ultraviolet divergences [1].

If a space has conic singularities, the complexity of the problem increases to a still greater extent, because the existence of delta-function singularities of R renders the use of the de Witt–Schwinger expansion impossible. For a locally planar space with one conic singularity, the problem can nevertheless be solved exactly [2, 3], because the presence of four Killing's vectors makes it possible to separate variables in the wave equation. But the introduction of additional singularities and, as a consequence, the loss of axial symmetry change the situation radically.

It appears that the way out can be found with the use of perturbation theory methods. In [3, 4], we reported some preliminary results obtained in consideration of nonlocal effects in multiconic spaces. The question of the behavior of high-order terms of the perturbation theory series and, in particular, its convergence, however, remains open.

Consider the Riemann space $E_{D-2} \times V_2$, which is the product of the $(D-2)$-dimensional Euclidean space and the two-dimensional Riemann surface. The length interval of such a space can be written as

$$ds^2 = \sum_{\mu=3}^{D} dx_\mu^2 + \mathrm{e}^{-\Omega(x_c)} \delta_{ab} dx_a dx_b \quad (a, b, c \ldots = 1, 2). \tag{1}$$

Here, we resorted to the fact that the two-dimensional Riemann surface is locally conformal to the Euclidean plane. In the coordinate system defined as above, the Poisson equation for the Green function on space (1) has the form

$$(\Delta_D + \widehat{V}) G_E = -\delta^{(D)}(x - x'), \tag{2}$$

where $\Delta_D = \sum_{\mu=3}^{D} \partial_\mu^2$ is the Laplace operator, and

$$\widehat{V}(x) = -F(x_c) \sum_{\mu=3}^{D} \partial_\mu^2, \qquad F(x_c) = 1 - \mathrm{e}^{-\Omega(x_c)}.$$

Consider the Nth term of the formal perturbation theory series for solving (2),

$$G = G_0 + G_0\widehat{V}G_0 + \ldots + G_0\widehat{V}G_0\ldots\widehat{V}G_0 + \ldots, \tag{3}$$

where G_0 is the Green function on the Euclidean D-dimensional space. Writing G_0 in the form of the Fourier integral yields

$$G_N(x,x') = \int \frac{d^2q_1}{(2\pi)^2}e^{iq_1x'}F(q_1)\ldots\int \frac{d^2q_N}{(2\pi)^2}e^{iq_Nx'}F(q_N)I_N^D(\{q\};x,x'), \tag{4}$$

where

$$I_N^D(\{q\};x,x') \equiv \int \frac{d^Dk}{(2\pi)^D}\frac{e^{ik(x-x')}(k_{\|}^2)^N}{k^2(k+Q_1)^2\cdot\ldots\cdot(k+Q_N)^2}, \tag{5}$$

$k_{\|}^2 = \sum\limits_{\mu=3}^{D} k_\mu^2$, $\mathbf{Q}_i = \sum_{j=1}^{i}\mathbf{q}_j$, $\mathbf{q}_j = (q_{1j}, q_{2j})$, and $F(q_i)$ is the Fourier transform of $F(x)$.

Note that with $D = 2$, $k_{\|}$ should be set equal to zero. This means that all terms of series (3) beginning with the second one vanish, and, if conformal coordinates are selected, the Green function on V_2 has the same form as the Green function on the Euclidean plane (also see [2, 4]).

In our approach, calculations of the regularized $G_{\rm reg}(x,x)$ value reduce to finding the regularized integral $I_N^D(\{q\};x,x) = I_N^D(\{q\})$. In terms of the integration over the Feynman parameters, $I_N^D(\{q\})$ can be represented as

$$I_N^D(\{q\}) = N!\int\frac{d^Dk}{(2\pi)^D}(k_{\|}^2)^N\int\limits_0^1 d\{z_n\}\frac{1}{[k^2z_0 + (k+Q_1)^2z_1 + \ldots + (k+Q_N)^2z_n]^{N+1}}, \tag{6}$$

where $d\{z_n\} = dz_0\ldots dz_N\cdot\delta\Big(1 - \sum\limits_{i=0}^{N} z_i\Big)$. Performing the $k\to k - \sum\limits_{i=0}^{N} z_iQ_i$ substitution of variables in (6) and integrating over angular variables transforms (6) to

$$I_N^D(\{q\}) = \int\limits_0^\infty\frac{dk_\perp k_\perp}{2\pi}\times\int\limits_0^\infty\frac{(k_{\|}^2)^{D/2-2+N}d(k_{\|}^2)}{\Gamma(D/2-1)(4\pi)^{D/2-1}}\times N!\int\limits_0^1 dz_1\int\limits_0^{1-z_1}dz_2\ldots\int\limits_0^{1-z_1-\ldots-z_{N-1}}dz_N\frac{1}{[k^2+A_N]^{N+1}}, \tag{7}$$

where

$$k_\perp = (k_1^2 + k_2^2)^{1/2}, \qquad A_N = \sum_{i=1}^{N} z_i(\mathbf{Q}_i)^2 - \Big(\sum_{i=1}^{N} z_i(\mathbf{Q}_i\Big)^2.$$

The further integration can conveniently be performed in terms of the new variables k and θ,

$$k^2 = k_\perp^2 + k_{\|}^2, \quad k\sin\theta = k_\perp, \quad k\cos\theta = k_{\|}.$$

After the integrations over θ, (7) becomes

$$I_N^D(\{q\}) = \frac{N!}{(2\pi)(4\pi)^{D/2-1}\Gamma(D/2-1)(D-2+2N)}\times\int\limits_0^1 dz_1\ldots\int\limits_0^{1-z_1-\ldots-z_{N-1}}dz_N\int\limits_0^\infty d(k^2)\frac{(k^2)^{D/2-1+N}}{(k^2+A_N)^{N+1}}.$$

The I_N^D integral written in this form can be calculated by the method of dimensional regularization. Indeed, at $D < 2$, the integral converges and can be transformed to

$$I_N^D(\{q\}) = -\frac{\Gamma(2-D/2)\Gamma(D/2-1+N)}{(4\pi)^{D/2}\Gamma(D/2)}\times\int\limits_0^1 dz_1\int\limits_0^{1-z_1}dz_2\ldots\int\limits_0^{1-z_1-\ldots-z_{N-1}}dz_N A_N^{D/2-1}. \tag{8}$$

This expression allows an analytic continuation into the region $D \geq 2$. At $D = 4, 6, \ldots$, the integral diverges because of the presence of poles at $\Gamma(2 - D/2)$. It follows that the singular part of integral (8) takes the form of the pole $(D-2k)^{-1}$, $k = 2, 3 \ldots$, and the finite contribution can be found by the technique usual for the method of dimensional regularization. Let us separately consider the case $D = 2$. Expression (8) is then finite and has an especially simple form

$$I_N(\{q\}) = -\frac{1}{4\pi N}. \tag{9}$$

The Nth term of series (3) at $D = 2$ takes the form

$$G_N(x, x) = -\frac{F^N(x)}{4\pi N}. \tag{10}$$

It follows that at $|F(x)| < 1$, the series is summable, and we obtain

$$G_{\text{reg}}(x, x) = \frac{1}{4\pi} \ln(1 - F(x)) = -\frac{\Omega(x)}{4\pi}, \tag{11}$$

which coincides with the result of [2, 4].

We see that with conformal coordinates selected on V_2, perturbation theory combined with the method of dimensional regularization leads to series with all terms well-defined. At $D = 2$, the series is summable, no matter what the form of the conformal factor, and the resulting expression for $G_{\text{reg}}(x, x)$ coincides with the one found by covariant separation of points.

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