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RADIOPHYSICS

STILL PARAMETRIC SOLITONS IN PERIODICALLY NONUNIFORM MEDIA WITH QUADRATIC NONLINEARITY

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Equations have been derived for the envelopes of still parametric solitons at the fundamental and double frequencies in periodically nonuniform media. The integrals of motion have been obtained and the exact solution which describes the spatial distribution of vibrations has been found. The conditions of formation and the properties of still solitons have been analyzed.

It is well known that envelope solitons can exist in dispersive media owing to nonlinear effects [1]. To date, the properties of solitons in cubically nonlinear media have been studied in sufficient detail. Of specific interest is the formation of slow solitons at frequencies that are close to the boundaries of nontransmitting band in periodically nonuniform structures. In particular, the propagation of vibrations in a forbidden frequency band is possible due to the nonlinear tunnel effect of slow solitons [2, 3].

In recent years, much attention is given to parametrically coupled solitons in media with quadratic nonlinearity. Their existence was predicted in [4, 5] and proved by optical experiments [6–8]. Optical parametric solitons have been the subject of many publications, including some reviews [8–11]. However, the issue of the formation and properties of slow parametric solitons in periodic structures with quadratic nonlinearity remains virtually unexplored to date. Parametric interaction of this type can be realized, in particular, if the frequencies of the fundamental and second harmonics are located on opposite sides of the bandwidth. The present article is devoted to the analysis of this important and interesting type of solitons.

Let us consider vibrations in a chain of circuits with identical resonance frequencies ω_0, asymmetric constants of linear coupling α_1, α_2, and different coefficients of quadratic nonlinearity γ_1, γ_2. Such vibrations are described by the equations

$$\begin{aligned} \ddot{x}_{2n} + \omega_0^2 x_{2n} + \gamma_2 x_{2n}^2 &= -\alpha_2\omega_0^2(2x_{2n} - x_{2n-1} - x_{2n+1}), \\ \ddot{x}_{2n+1} + \omega_0^2 x_{2n+1} + \gamma_1 x_{2n+1}^2 &= -\alpha_1\omega_0^2(2x_{2n+1} - x_{2n} - x_{2n+2}). \end{aligned} \tag{1}$$

In a linear approximation holds the dispersion equation

$$(\omega/\omega_0)^2 = 1 + (\alpha_2 + \alpha_1) \pm \sqrt{(\alpha_2 - \alpha_1)^2 - 4\alpha_2\alpha_1 \sin^2 ka},$$

which relates the frequency of vibrations ω and a wave number k (Fig. 1).

The dispersion curve shows the chain transmission and nontransmission regions. In the case $\alpha_1 < \alpha_2$ the lower branch is located in the interval $\omega_0 < \omega < \omega_c$, where $\omega_c = \omega_0(1 + 2\alpha_1)^{1/2}$, and the upper one is located in $\omega_b < \omega < \omega_a$, where $\omega_b = \omega_0(1 + 2\alpha_2)^{1/2}$ and $\omega_a = \omega_0(1 + 2\alpha_1 + 2\alpha_2)^{1/2}$.

In a parametric soliton, the first and second harmonics are coupled [4–5]. For a still soliton to be excited, it is necessary that the second harmonic have a frequency $\omega_2 \cong \omega_a$ and a wave number $k_2 = 0$, and the first harmonic consist of two opposite waves at the frequency $\omega_1 = \omega_2/2$ and with wave vectors $k_{11} = -k_{12}$ (see Fig. 1).

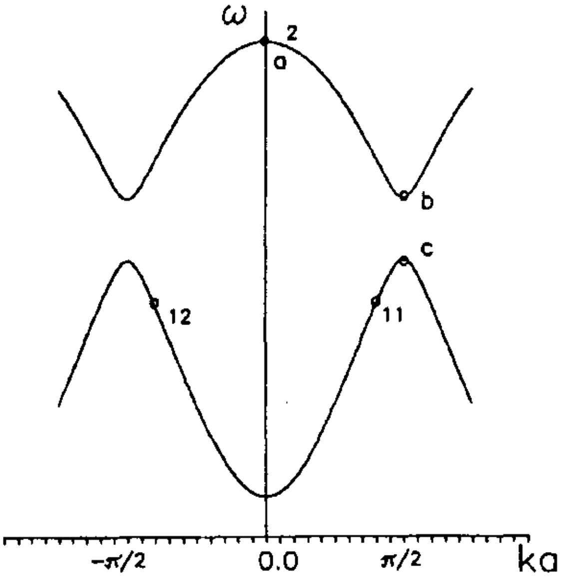

Fig. 1
Dispersion curve for a system of alternating coupled vibrating systems. Two waves of fundamental frequency 11 and 12 with opposite wave vectors excite a still double-frequency standing wave 2.

Let us now discuss the parametric interaction of the wave packets. For this purpose we set

$$x_{2n} = 0.5[b_{11}\exp(i\omega_1 t - 2nik_{11}a) + b_{12}\exp(i\omega_1 t - 2nik_{12}a) + b_2\exp(i\omega_2 t)] + \text{c.c.},$$
$$x_{2n+1} = 0.5[B_{11}\exp(i\omega_1 t - (2n+1)ik_{11}a) + B_{12}\exp(i\omega_1 t - (2n+1)ik_{12}a) + B_2\exp(i\omega_2 t)] + \text{c.c.} \quad (2)$$

Here b, B are six slowly varying with time complex amplitudes of vibrations at the two frequencies in even and odd circuits, respectively. For weakly nonlinear vibrations it is possible to find the relations between the amplitudes b and B in adjacent circuits at each frequency. Omitting the calculations, we present the final formulas for nonlocal interrelation of the amplitudes in the linear approximation at frequency ω_2

$$b_2 = -(\alpha_2/\alpha_1)B_2 + [\alpha_2/(\alpha_1+\alpha_2)]a^2\frac{\partial^2 B_2}{\partial z^2},$$
$$B_2 = -(\alpha_1/\alpha_2)b_2 + [\alpha_1/(\alpha_1+\alpha_2)]a^2\frac{\partial^2 b_2}{\partial z^2}. \quad (3)$$

At frequency ω_1 the vibration amplitudes in adjacent circuits are related as

$$b_1 = -\mu_1 B_1 + \mu_2\frac{\partial B_1}{\partial z} + \mu_3\frac{\partial^2 B_1}{\partial z^2},$$
$$B_1 = -\eta_1 b_1 + \eta_2\frac{\partial b_1}{\partial z} + \eta_3\frac{\partial^2 b_1}{\partial z^2},$$

where μ and η are constants expressed in an intricate manner in terms of the periodic system parameters.

Substituting (2) in (1) with regard to (3), we can obtain the following equations for the slowly varying amplitudes of nonpropagating waves at the fundamental and double-frequencies

$$\frac{\partial B_2}{\partial t} = iD_2\frac{\partial^2 B_2}{\partial z^2} + i\beta_2 B_{11}B_{12},$$
$$\frac{\partial B_{11}}{\partial t} + u_1\frac{\partial B_{11}}{\partial z} = iD_1\frac{\partial^2 B_{11}}{\partial z^2} + i\beta_1 B_{12}^* B_2, \quad (4)$$
$$\frac{\partial B_{12}}{\partial t} - u_1\frac{\partial B_{12}}{\partial z} = iD_1\frac{\partial^2 B_{12}}{\partial z^2} + i\beta_1 B_{11}^* B_2,$$

where u_1 is the group velocity at the fundamental frequency ω_1, D is the dispersion coefficient, and β is the coefficient of nonlinearity.

Similar equations were studied in [9–12] for the case of three-wave interaction in a dispersive quadratically nonlinear medium. However, in the present case we deal with spatial localization of the solitons. Equations (4) have an integral of motion I_3 (see [9, 10]) in the form

$$I_3 = \int \left[\frac{D_1}{\beta_1} \left| \frac{\partial B_{11}}{\partial z} \right|^2 + \frac{D_1}{\beta_1} \left| \frac{\partial B_{12}}{\partial z} \right|^2 + \frac{D_2}{\beta_2} \left| \frac{\partial B_2}{\partial z} \right|^2 - B_{11}B_{12}B_2^* - B_{11}^*B_{12}^*B_2 + \frac{iu_1}{2\beta_1} \left(B_{11}\frac{\partial B_{11}^*}{\partial z} + B_{11}^*\frac{\partial B_{11}}{\partial z} - B_{12}\frac{\partial B_{12}^*}{\partial z} - B_{12}^*\frac{\partial B_{12}}{\partial z} \right) \right] \partial z.$$

The sign of I_3 indicates the presence or absence of a soliton. It should be pointed out that to (4) also correspond an energy conservation law and another integral I_2 which is written as

$$I_2 = \int \left(B_{11}\frac{\partial B_{11}^*}{\partial z} + B_{11}^*\frac{\partial B_{11}}{\partial z} - B_{12}\frac{\partial B_{12}^*}{\partial z} - B_{12}^*\frac{\partial B_{12}}{\partial z} \right) \partial z.$$

These integrals allow one to check the correctness of numerical calculations.

A spatially localized solution to (4) is sought in the form $B_2 = A_2(z)\exp(i\Omega_2 t)$, $B_{11} = A_{11}(z)\exp(i\Omega_{11}t - iq_{11}z)$, and $B_{12} = A_{12}(z)\exp(i\Omega_{12}t + iq_{12}z)$. Substituting these expressions in (4), we obtain the equations for the envelopes of parametrically coupled solitons

$$\begin{aligned} D_2A_2'' &= \Omega A_2 - \beta_2 A_{11}A_{12}, \\ D_1A_{11}'' &= (\Omega_{11} - u_1q_{11} + D_1q_{11}^2)A_{11} - \beta_1 A_2A_{12}, \\ D_1A_{12}'' &= (\Omega_{12} - u_1q_{12} + D_1q_{12}^2)A_{12} - \beta_1 A_2A_{11}. \end{aligned} \tag{5}$$

The frequency detunings and wave numbers are related by the formulas $\Omega_{11} + \Omega_{12} = \Omega_2$ and $q_{11} = q_{12} = u_1/2D_1$. Equations (5) have an exact analytical solution in the form of still solitons, $A = a\,\mathrm{sech}^2(z/l)$. Substituting the soliton solution in (5), we obtain the following relationships between the soliton parameters and the equation constants

$$a_{11} = a_{12} = (6/l^2)\sqrt{D_1D_2/\beta_1\beta_2}, \quad a_2 = 6D_1/l^2\beta_1, \tag{6}$$

$$\Omega_2 = 4D_2/l^2, \quad \Omega_{11} = \Omega_{12} = 2D_2/l^2, \tag{7}$$

$$l^2 = 8D_1(2D_1 - D_2)/u_1^2. \tag{8}$$

Thus, the waves of the fundamental frequency form a standing wave. We see that D_1 and D_2 must have the same sign (see (6)). Analyzing (8), we further find that parametric solitons of this type can be formed in a medium with $D_2/D_1 = 2$ if $u_i = 0$. The spatial extent of such solitons, l, depends on the amplitude in accordance with (6). If $D_2/D_1 < 2$, then l^2 assumes a fixed value (8), determined by the parameters D and u_1. A typical spatial distribution of still soliton amplitudes is shown in Fig. 2. The soliton existence condition $D_2/D_1 \le 2$ imposes restrictions on the coupling coefficients α. For instance, in the case $u_1 = 0$ we have $2\alpha_1^2(\alpha_2 + \alpha_1) = \alpha_2(\alpha_2 - \alpha_1)^2$ which implies that $\alpha_2 \cong 2.66\alpha_1$. Note that the detunings of both fundamental-frequency waves (7) and the additions to the wave numbers are equal to each other. For the formation of a nonuniform standing wave at the fundamental frequency, set of equations (5) is reduced to two equations for which localized solutions were found by numerical and variational techniques in a wide range of parameter variation [13].

In conclusion, let us formulate the main results. We have investigated the process of formation of still parametric solitons in periodic systems in which the second harmonic is excited by two opposite waves. Reduced equations that allow a soliton solution have been derived using the example of a chain of alternating circuits. Relationships between the soliton parameters have been obtained. The conditions under which the reduced equations can have a soliton solution have been found.

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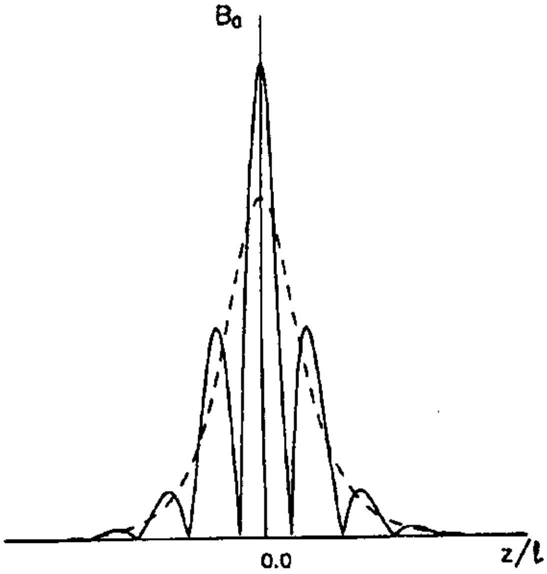

Fig. 2

Spatial distribution of amplitudes for a still parametric soliton at the fundamental (solid curve) and double- (dashed curve) frequencies.

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