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THEORETICAL AND MATHEMATICAL PHYSICS

FOUNDATIONS OF POSSIBILITY THEORY. METHODS OF OPTIMAL ESTIMATION AND DECISION MAKING 4. MAXIMAL EXTENSION OF POSSIBILITY

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It is shown that a possibility $P(\cdot):\ \mathcal{A} \to [0,1]$ can always be extended from an arbitrary σ-algebra $\mathcal{A}$ of subsets of X to the algebra $\mathcal{P}(X)$ of all subsets of X with preservation of all properties of the possibility, and that a measure $p(\cdot):\ \mathcal{L}(X) \to [0,1]$ specifying a possibility of fuzzy events can be extended with preservation of its properties to the class of all functions $X \to [0,1]$.

INTRODUCTION

Countability and measurability, which are fundamental mathematical notions determining applicability of mathematical probability theory to modeling of reality [1], play a substantially less significant role in possibility theory. It is shown below that a possibility $P(\cdot):\ \mathcal{A} \to [0,1]$ can always be extended from an arbitrary σ-algebra $\mathcal{A}$ of subsets of X to the algebra $\mathcal{P}(X)$ of all subsets of X with preservation of all its properties, and a measure $p(\cdot):\ \mathcal{L}(X) \to [0,1]$ specifying a possibility of fuzzy events can be extended with preservation of properties to the class of all functions $X \to [0,1]$. This means that, in possibility theory, any subset of X can be considered an event (measurable set) and assigned a possibility: an arbitrary function $\mu(\cdot):\ x \to [0,1]$ can be considered the characteristic function of the corresponding fuzzy event, and the value $p(\mu(\cdot))$ its possibility. As distinct from probability theory, in possibility theory, any, including uncountable, unions and intersections of events are events; in this respect, possibility theory is simpler than probability theory. However, as has already been mentioned in [2], the cost of this simplification is the loss of continuity: a possibility, being completely additive, is not a continuous function on $\mathcal{P}(X)$.

In this work, we construct a (maximal) extension of a possibility $P(\cdot)$ to the algebra $\mathcal{P}(X)$ and of a measure $p(\cdot)$ to the class of all functions $X \to [0,1]$. In what follows, "Definition 4.2" means "Definition 2 from [4]", a reference to formula (2.7) is a reference to formula (7) from [2], etc.

1. EXTENSION OF A POSSIBILITY TO THE ALGEBRA $\mathcal{P}(X)$ OF ALL SUBSETS OF X

There are at least two circumstances witnessing that the probability-theoretic scheme is not completely adequate in the possibility theory under consideration. First, unlike a probability, a countable-additive possibility is in general not continuous with respect to convergence of the sequence of events defined in item 3 of Theorem 2.2. Alternatively, the requirement of countable additivity not only fails to ensure the continuity of a possibility, but is also unnatural, because, in the case under consideration, the addition operation is defined so that it is possible to "add" any set of "terms" which need not be finite or countable.

Let us show that, in the theory under consideration, a possibility can always be extended to the algebra $\mathcal{P}(X)$ of all subsets of X with preservation of all its properties and can be specified by a distribution. For this purpose, for each $p \in [\alpha, 1]$, where $\alpha = \inf_{A \in \mathcal{A}, A \neq \varnothing} P(A)$, we define the set

$$S_p = \bigcap_{\substack{A \in \mathcal{A}, \\ P(X \setminus A) \leq p}} A = X \setminus \bigcup_{\substack{A \in \mathcal{A}, \\ P(A) \leq p}} A. \tag{1}$$

If $\alpha > 0$, then only the empty set $\varnothing \in \mathcal{A}$ satisfies the condition $P(A) < \alpha$ with $A \in \mathcal{A}$. It is then natural to extend the definition of S_p to $p \in [0, \alpha)$ by setting $S_p = X$. The sets S_p with $0 \leq p \leq 1$, possibly nonmeasurable, form a monotone family in the sense that if $0 \leq p \leq q \leq 1$, then $S_p \supset S_q$, and it is assumed that $S_p|_{p=1} = \varnothing$ and $S_p|_{p=0} = X$. In what follows, it is assumed that $P(\varnothing) = 0$.

Definition 1. Suppose that $B \subset X$ is an arbitrary set and $\mathcal{D}(B) = \{p \in [0,1], S_p \cap B \neq \varnothing\}$. Put

$$\overline{P}(B) = \begin{cases} \sup \mathcal{D}(B) & \text{if} \quad \mathcal{D}(B) \neq \varnothing, \\ 0 & \text{if} \quad \mathcal{D}(B) = \varnothing, \quad B \subset X \end{cases} \tag{2}$$

and

$$\varphi(x) = \overline{P}(\{x\}), \ x \in X. \tag{3}$$

Since $\mathcal{D}(B) = \bigcup_{x \in B} \mathcal{D}(\{x\}) = \bigcup_{x \in B} \{p \in [0,1], \ x \in S_p\}$, we have

$$\overline{P}(B) = \begin{cases} \sup_{x \in B} \varphi(x) & \text{if} \quad B \neq \varnothing, \\ 0 & \text{if} \quad B = \varnothing, \quad B \subset X. \end{cases} \tag{2*}$$

Let us show that the function $\overline{P}(\cdot)$ of sets extends the possibility $P(\cdot)$ from the σ-algebra $\mathcal{A}$ to the algebra $\mathcal{P}(X)$ of all subsets of X; according to (2*), the function $\varphi(\cdot)$ defined by (3) specifies the distribution of the possibility so extended.

Theorem 1. 1. *$\overline{P}(B)$, where $B \subset X$, is a possibility on the algebra $\mathcal{P}(X)$ of all subsets of X; i.e., for any $A, B \in \mathcal{P}(X)$,*

$$A \subset B \Rightarrow \overline{P}(A) \leq \overline{P}(B) \qquad (\textit{monotonicity}),$$

$$\overline{P}(A \cup B) = \max(\overline{P}(A), \overline{P}(B)) \qquad (\textit{additivity}).$$

For any family $A_j \in \mathcal{P}(X)$, $j \in J$, $\overline{P}(\bigcup_{j \in J} A_j) = \sup_{j \in J} \overline{P}(A_j)$.

2. *For any $B \in \mathcal{P}(X)$ the possibility $\overline{P}(B)$ is determined according to formulas (2*) and (3) by its distribution $\varphi(x)$ specified by (3) for $x \in X$, i.e., by its values on singletons from X.*

3. *For any $A \in \mathcal{A}$, $\overline{P}(A) = P(A)$.*

Proof. 1. If $A \subset B$, then, obviously, $\mathcal{D}(A) \subset \mathcal{D}(B)$ and, therefore, $\overline{P}(A) \leq \overline{P}(B)$. Then, since $\mathcal{D}(A \cup B) = \mathcal{D}(A) \cup \mathcal{D}(B)$, it follows that $\overline{P}(A \cup B) = \sup \mathcal{D}(A \cup B) = \max(\sup \mathcal{D}(A), \sup \mathcal{D}(B)) = \max(\overline{P}(A), \overline{P}(B))$. Finally,

$$\overline{P}(\bigcup_{j \in J} A_j) = \sup \mathcal{D}(\bigcup_{j \in I} A_j) = \sup_{j \in I} \sup \mathcal{D}(A_j) = \sup_{j \in I} \overline{P}(A_j).$$

2. This assertion follows from (2), (3), and (2*).

3. Suppose that $A \in \mathcal{A}$ and $P(A) = q > 0$. If $p \geq q$, then, according to (2), $S_p \subset X \setminus A$ and, therefore, $S_p \cap A = \varnothing$. It follows that

$$\overline{P}(A) = \sup\{p | \, S_p \cap A \neq \varnothing\} \leq q = P(A). \tag{3*}$$

On the other hand, if $p < q$, then $S_p \cap A \neq \varnothing$ because $S_p = X \setminus \bigcup_{P(B) \leq p} B$ and $A \not\subset \bigcup_{P(B) \leq p} B$. Let $p_1 \leq p_2 \leq \dots \leq p_n \leq \dots$, $p_j < q$, $j = 1, 2, \dots$, and $q = \lim_{j \to \infty} p_j$. Then $\overline{P}(A) = \sup\{p \mid S_p \cap A \neq \varnothing\} \geq \sup_{1 \leq j \leq \infty} \{p_j \mid S_{p_j} \cap A \neq \varnothing\} = q = P(A)$. If $P(A) = q = 0$, then $\overline{P}(A) = 0$ by (3*). ■

Remark 1. Let X be a locally compact Hausdorff topological space and $\varphi(\cdot)\colon X \to [0,1]$ be an upper semicontinuous function (distribution of possibility). Then $P(\cdot)$ defined as $\overline{P}(A) = \sup_{x \in A} \varphi(x)$ for nonempty

$A \in \mathcal{P}(X)$ and vanishing at the empty set ($P(\varnothing) = 0$) is a Choquet capacity (see, e.g., [3]). Indeed, $\overline{P}(\cdot)$ is a Choquet capacity if the following conditions are fulfilled:

(i) If $A_1, A_2 \in \mathcal{P}(X)$ and $A_1 \subset A_2$, then $\overline{P}(A_1) \le \overline{P}(A_2)$;

(ii) If $A_n \in \mathcal{P}(X)$ for $n = 1, 2, \ldots$, $A_1 \subset A_2 \subset \ldots$, and $A = \bigcup_n A_n$, then $\overline{P}(A_n) \uparrow \overline{P}(A)$ as $n \to \infty$;

(iii) If K_n are compact subsets of X for $n = 1, 2, \ldots$, $K_1 \supset K_2 \supset \ldots$, and $K = \bigcap_n K_n$, then $\overline{P}(K_n) \downarrow \overline{P}(K)$ as $n \to \infty$.

According to Theorem 1, $\overline{P}(\cdot)$ has properties (i) and (ii). Let us show that $\sup_{x \in K_n} \varphi(x) \downarrow \sup_{x \in K} \varphi(x)$. Since $\varphi(\cdot)$ is upper semicontinuous and K_n are compact, the set $K_{*n} = \{x \in K_n,\ \varphi(x) = \sup_{y \in K_n} \varphi(y)\}$ is nonempty and compact for any $n = 1, 2, \ldots$. Suppose that $x_i \in K_{*i}$ for $i = 1, 2, \ldots$ and $\{x_{i_n}\}$ is a convergent subsequence of $\{x_i\} \subset K_1$ with $\mathring{x} = \lim_{n \to \infty} x_{i_n}$. Since $\varphi(x_1) \ge \varphi(x_2) \ge \ldots$, the sequence $\{\varphi(x_n)\}$ converges, and $\lim_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} \varphi(x_{i_n}) = \varphi(\mathring{x})$. Indeed, on the one hand, $\varphi(x_{i_n}) \ge \varphi(\mathring{x})$ for $n = 1, 2, \ldots$; on the other hand, $\lim_{n \to \infty} \varphi(x_{i_n}) \le \varphi(\mathring{x})$ because $\varphi(\cdot)$ is upper semicontinuous. As $\varphi(x_{i_n}) \ge \sup_{x \in K} \varphi(x)$ for $n = 1, 2, \ldots$, we have $\varphi(\mathring{x}) \ge \sup_{x \in K} \varphi(x)$, and as $\mathring{x} \in K_{i_n}$ for $n = 1, 2, \ldots$, we have $\mathring{x} \in K$; hence $\varphi(\mathring{x}) \le \sup_{x \in K} \varphi(x)$. Therefore, $\sup_{x \in K} \varphi(x) = \varphi(\mathring{x}) = \lim_{n \to \infty} \sup_{x \in K_n} \varphi(x)$. ■

The family S_p, $0 \le p \le 1$, specified by (1) determines a distribution $\varphi(x)$ for $x \in X$ according to (3). Let us consider to which degree the distribution $\varphi(\cdot)$ determines initial family (1). In what follows, we assume that $\sup\{p \in [0,1] \mid p \in \varnothing\} = 0$ and use the following expression for the possibility $\overline{P}(\cdot)$: $\overline{P}(A) = \sup \mathcal{D}(A) = \sup\{p \in [0,1] \mid S_p \cap A \ne \varnothing\}$ for any $A \in \mathcal{P}(X)$.

Lemma 1. 1. *For any* $p \in [0,1]$, $S_{\sim p} = \{x \in X, \varphi(x) > p\} \subset S_p \subset \{x \in X, \varphi(x) \ge p\} = S_p^{\sim}$.

2. *For any* $A \in \mathcal{P}(X)$, $P_{\sim}(A) = \sup\{p \in [0,1] \mid S_{\sim p} \cap A \ne \varnothing\} = \overline{P}(A) = \sup\{p \in [0,1] \mid S_p^{\sim} \cap A \ne \varnothing\} = P^{\sim}(A)$.

Proof. 1. For $x \in S_p$, we have $\varphi(x) = \sup\{q, x \in S_q\} \ge p$; therefore, $S_p \subset S_p^{\sim}$. If $x \in S_{\sim p}$, then $\sup\{q | x \in S_q\} = \varphi(x) > p$. Hence, there exists $\varepsilon > 0$ such that $q_\varepsilon = \varphi(x) - \varepsilon > p$ and $x \in S_{q_\varepsilon}$. Since $S_{q_\varepsilon} \subset S_p$, we have $x \in S_p$, i.e., $S_{\sim p} \subset S_p$.

2. We have $\varphi^{\sim}(x) \triangleq \sup\{p \mid x \in S_p^{\sim}\} = \sup\{p \mid \varphi(x) \ge p\} = \varphi(x) = \sup\{p \mid \varphi(x) > p\} = \sup\{p,\ x \in S_{\sim p}\} \triangleq \varphi_{\sim}(x)$ for $x \in X$; therefore, $P_{\sim}(A) = \sup_{x \in A} \varphi_{\sim}(x) = \overline{P}(A) = \sup_{x \in A} \varphi^{\sim}(x) = P^{\sim}(A)$ for $A \in \mathcal{P}(X)$. ■

Remark 2. The family $\overline{S}_p$ with $p \in [0,1]$ defined with the use of the possibility $\overline{P}(\cdot)$ by the formula $\overline{S}_p = X \setminus \bigcup_{A \in \mathcal{P}(X),\ \overline{P}(A) \le p} A$ for $0 \le p \le 1$, which is similar to (1), gives the same values of the possibility if S_p is replaced by $\overline{S}_p$ in (2). Indeed, obviously, $\overline{S}_p \subset S_p$ because $\mathcal{A} \subset \mathcal{P}(X)$ and $\overline{S}_p = X \setminus \bigcup_{A \in \mathcal{P}(X),\ x \in A, \sup \varphi(x) \le p} =$ $= X \setminus \{x \in X,\ \varphi(x) \le p\} = \{x \in X,\ \varphi(x) > p\} = S_{\sim p}$. Therefore, $S_{\sim p} \subset \overline{S}_p \subset S_p \subset S_p^{\sim}$, and, according to Lemma 1, the family $\overline{S}_p$ with $0 \le p \le 1$ in (2) gives the same values of $\overline{P}(\cdot)$. ■

2. UNIQUENESS OF EXTENSION OF A POSSIBILITY

Generally, a possibility $p(\cdot)$ admits different extensions to $\mathcal{P}(X)$. Indeed, suppose, for example, that $X = \bigcup_{j=1}^{\infty} A_j$, where $A_i \cap A_j = \varnothing$ for $i \ne j$, $i, j = 1, 2, \ldots$, and $\mathcal{A}$ is the minimal σ-algebra containing $\{A_j\}$. An extension of $P(\cdot)$ to $\mathcal{P}(X)$ is an arbitrary possibility $\widehat{P}(\cdot)$ satisfying the condition

$$\widehat{P}(A_j) = \sup_{x \in A_j} \hat{\varphi}(x) = P(A_j), \qquad j = 1, 2, \ldots, \tag{4}$$

where $\hat{\varphi}(\cdot)\colon X \to [0,1]$ is the distribution of $\widehat{P}(\cdot)$. If A_j is not a singleton from X, then condition (4) does not determine the distribution $\hat{\varphi}(x)$ for $x \in A_j$ uniquely for $j = 1, 2, \ldots$.

The extension $\overline{P}(\cdot)$ developed above can be called maximal in the sense that

$$\overline{P}(B) \geq \widehat{P}(B), \qquad B \in \mathcal{P}(X)$$

for any other extension $\widehat{P}(\cdot)$. To prove this, it is useful to consider another representation of the distribution of $\overline{P}(\cdot)$, which is also of interest of its own.

Lemma 2. *Put*

$$\varphi_*(x) = \inf\{P(A),\ A \in \mathcal{A}, \quad x \in A\}, \quad x \in X. \tag{5}$$

Then $\varphi_*(x) = \varphi(x)$ for $x \in X$, where the distribution $\varphi(\cdot)$ is defined by (3) *and, therefore,*

$$P_*(B) = \sup_{x \in B} \varphi_*(x) = \overline{P}(B)$$

for any nonempty $B \in \mathcal{P}(X)$.

Proof. Let us note $x \in X$ and take an arbitrary $\varepsilon > 0$. According to (5), first, $P(A) > \varphi_*(x) - \varepsilon = p_*$ for any $A \in \mathcal{A}$ containing x and, second, there exists $A_\varepsilon \in \mathcal{A}$ such that it contains x and $P(A_\varepsilon) \leq \varphi_*(x) + \varepsilon = p^*$. According to definition (1), $x \in S_{p_*}$ because $x \notin \bigcup_{\substack{A \in \mathcal{A} \\ P(A) \leq p_*}} A$, and $x \notin S_{p^*}$ because $x \in \bigcup_{\substack{A \in \mathcal{A} \\ P(A) \leq p^* A}} A$. Since $\varphi(x) = \sup\{p \in [0,1] \mid x \in S_p\}$, we have $\varphi_*(x) - \varepsilon \leq \varphi(x) \leq \varphi_*(x) + \varepsilon$, and $\varphi_*(x) = \varphi(x)$ for $x \in X$ because $\varepsilon > 0$ is arbitrary. ■

Lemma 3. *Equality* (2) *determines a maximal extension $\overline{P}(\cdot)$ of the possibility $P(\cdot)$ in the sense that $\overline{P}(B) \geq \widehat{P}(B)$, where $B \in \mathcal{P}(X)$, for any other extension $\widehat{P}(\cdot)$.*

Proof. Let $\hat{\varphi}(x) = \widehat{P}(\{x\})$ for $x \in X$ be the distribution of $\widehat{P}(\cdot)$. We show that $\hat{\varphi}(x) \leq \varphi(x)$ for $x \in X$. If a singleton $\{\mathring{x}\} \in \mathcal{A}$, then $\hat{\varphi}(\mathring{x}) = \varphi(\mathring{x})$; if $\{\mathring{x}\} \notin \mathcal{A}$, then we consider a minimal $A = \mathring{A} \in \mathcal{A}$ containing $\mathring{x}$. In other words, $\mathring{x} \in \mathring{A} \in \mathcal{A}$, and, for any $\widetilde{A} \in \mathcal{A}$ such that $\mathring{x} \in \widetilde{A}$, $\mathring{A} \subset \widetilde{A}$. Since $P(\cdot)$ is monotone, we then have

$$\varphi(\mathring{x}) = \varphi_*(\mathring{x}) = \inf\{P(A),\ A \in \mathcal{A},\ \mathring{x} \in A\} = P(\mathring{A}).$$

For any other point $x \in \mathring{A}$, $\varphi(x) = \varphi_*(x) = \varphi(\mathring{x})$ because $\mathring{A}$ is minimal for each point thereof. As $P(\mathring{A}) = \sup_{x \in \mathring{A}} \hat{\varphi}(x)$, we have $\hat{\varphi}(x) \leq \varphi(x)$ for $x \in \mathring{A}$. ■

3. EXTENSION OF A POSSIBILITY OF FUZZY EVENTS

In conclusion, we consider extension of a possibility $p(f(\cdot))$ of fuzzy events specified by their characteristic functions $f(\cdot) \in \mathcal{L}(X)$. Let $\overline{\mathcal{L}}(X)$ be the class of all functions $f(\cdot)$ defined on X and taking on values in $\mathcal{R}_{(p)}$ where addition and multiplication operations (2.3) are defined.

Definition 2. A function $\overline{p}(\cdot)$ defined on $\overline{\mathcal{L}}(X)$ and taking on values in $\mathcal{R}_{(p)}$ is called the measure if it is linear in the sense of Definition 2.1 and, for any family* $f_j(\cdot) \in \overline{\mathcal{L}}(X)$ with $j \in J$,

$$\overline{p}(\sup_{j \in J} f_j(\cdot)) = \sup_{j \in J} \overline{p}(f_j(\cdot)). \tag{6}$$

We say that the measure $\overline{p}(f(\cdot))$ for $f(\cdot) \in \overline{\mathcal{L}}(X)$ is a maximal extension of the measure $p(f(\cdot))$ for $f(\cdot) \in \mathcal{L}(X)$ if $\overline{p}(f(\cdot)) = p(f(\cdot))$ for $f(\cdot) \in \mathcal{L}(X)$ and, for any other extension $\hat{p}(\cdot)$,

$$\overline{p}(f(\cdot)) \geq \hat{p}(f(\cdot)),\ f \in \overline{\mathcal{L}}(X).$$

* By linearity of (2.7), $\overline{p}(\cdot)$ does not decrease monotonically (2.9); therefore, $\overline{p}(\inf_{j \in J} f_j(\cdot)) \leq \overline{p}(f_i(\cdot)) \leq \overline{p}(f_i(\cdot)) \leq \overline{p}(\sup_{j \in J} f_j(\cdot))$, where $i \in J$. Thus, generally, $\overline{p}(\inf_{j \in J} f_j(\cdot)) \leq \inf_{j \in J} \overline{p}(f_j(\cdot)) \leq \sup_{j \in J} \overline{p}(f_j(\cdot)) \leq \overline{p}(\sup_{j \in J} f_j(\cdot))$.

Theorem 2. 1. *An arbitrary measure $\overline{p}(\cdot)$ has the representation*

$$\overline{p}(f(\cdot)) = \sup_{x\in X} \min(f(x), \varphi(x)),\ f(\cdot) \in \overline{\mathcal{L}}(X), \tag{7}$$

where $\varphi(\cdot) \in \overline{\mathcal{L}}(X)$ is the distribution of the possibility $\overline{P}(\cdot)$,

$$\overline{P}(A) = \overline{p}(\chi_A(\cdot)) = \sup_{x\in A} \varphi(x),\ A \in \mathcal{P}(X),$$

and

$$\varphi(x) = \overline{p}(\delta_x(\cdot)),\ x \in X,$$

where $\{\delta_y(\cdot),\ y \in X\}$ is the family of the functions

$$\delta_y(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}\ x \in X,\ y \in X,$$

from $\overline{\mathcal{L}}(X)$.

2. *An arbitrary function $f(\cdot) \in \overline{\mathcal{L}}(X)$ is integrable with respect to the possibility $\overline{P}(\cdot)$ in the sense of Definition 4.2, $\overline{p}(f(\cdot))$ is its integral, and $\overline{P}(A) = \overline{p}(\chi_A(\cdot))$ for $A \in \mathcal{P}(X)$; thus, there is a one-to-one correspondence between $\overline{P}(A)$, $A \in \mathcal{P}(X)$, and $\overline{p}(f(\cdot))$, $f(\cdot) \in \overline{\mathcal{L}}(X)$.*

3. *$\overline{P}(\cdot)$ is a maximal extension of $P(\cdot)$ if and only if $\overline{p}(\cdot)$ is a maximal extension of $p(\cdot)$.*

Proof. 1. For $f(\cdot) \in \overline{\mathcal{L}}(X)$, the "integral representation"

$$f(x) = \sup_{y\in X} \min(\delta_y(x), f(y)) \qquad \text{for } x \in X$$

holds. By (6) and the linearity of $\overline{p}(\cdot)$,

$$\overline{p}(f(\cdot)) = \sup_{y\in X} \overline{p}(\min(\delta_y(\cdot), f(y))) = \sup_{y\in X} \min(f(y), \overline{p}(\delta_y(\cdot))) = \sup_{y\in X} \min(f(y), \varphi(y)).$$

The proof of assertion 2 of Theorem 2 is the same as the proof of Theorem 4.3. Assertion 3 is evident. ■

Theorem 2 specifies a maximal extension $\overline{p}(\cdot)$ of a possibility $p(\cdot)$ of fuzzy events specified by their characteristic functions from $\mathcal{L}(X)$ to the class of all fuzzy events with characteristic functions from the class $\overline{\mathcal{L}}(X)$ and gives a representation of the extended possibility $\overline{p}(\cdot)$ in the form of integral (7) of the characteristic function of a fuzzy event $f(\cdot) \in \overline{\mathcal{L}}(X)$ with respect to the possibility $\overline{P}(\cdot)$.

Remark 3. Any function $f(\cdot) \in \mathcal{L}(X)$ has the representation

$$f(x) = \sup_{0\leq\alpha\leq 1} \min(\lambda, \chi_\alpha(x)) = \sup_{0\leq\alpha\leq 1} \min(\alpha, \tilde{\chi}_\alpha(x)),\ x \in X,$$

where $\chi_\alpha(\cdot)$ and $\tilde{\chi}_\alpha(\cdot)$ are the characteristic functions of the sets $A_\alpha = \{x \in X,\ f(x) = \alpha\}$ and $\tilde{A}_\alpha = \{x \in X,\ f(x) \geq \alpha\}$, respectively. Property (6) of a measure and linearity (2.7) of $\overline{p}(\cdot)$ imply that

$$\overline{p}(f(\cdot)) = \sup_{0\leq\alpha\leq 1} \min(\alpha, \overline{P}(A_\alpha)) = \sup_{0\leq\alpha\leq 1} \min(\alpha, \overline{P}(\tilde{A}_\alpha)).$$

The latter expression is known as the Sugeno integral which is defined in [5] on the class of (measurable) functions from $\mathcal{L}(X)$.

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