Moscow University Physics Bulletin
Vol. 53, No. 1, pp. 14–18, 1998

Vestnik Moskovskogo Universiteta. Fizika
UDC 539.171

PARTICULAR SOLUTIONS TO NONLINEAR SCHRÖDINGER-TYPE EQUATIONS OF SPECIAL FORM

E. G. Barinova, A. G. Kornienko, A. R. Frenkin, and G. A. Chizhov

Nonexponential solutions to one-dimensional nonlinear Schrödinger-type equations of special form derived by the quadrature method are studied. For the first time examples of analytic solutions to some types of these equations are given.

Solutions to nonlinear Schrödinger-type equations (NSE) of the form

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + F(|\Psi|)\Psi$$

were discussed in [1, 2]. Particular solutions of such nonlinear equations are described in many publications, including reviews [3, 4]. It should be pointed out that three clearly defined approaches are used for studying nonlinear equations of this type:

(a) investigating nonlinear effects in one-dimensional systems which are produced from the NSE in the search for solutions of special form;

(b) finding solutions by the inverse problem method;

(c) finding integral transformations which reduce the NSE to an algebraic problem or to a known class of differential equations.

In the present paper, as in [1, 2], we follow the first approach. This is justified by the possibility of reducing the NSE to a one-dimensional differential equation. The structure of the equation that arises in this approach is similar to those that appear during the solution of one-dimensional problems in classical mechanics. Despite the apparent simplicity of the equations produced and the existence of a complete mathematical theory that allows one to make some general statements on the character of their solutions [5], we describe here some physically interesting examples. Such problems are important because some recent models include polynomial potentials of degree higher than the traditionally employed fourth degree, or logarithmic factors. They might arise in solid state physics when defects of various nature are taken into account [6], and also in advanced supersymmetrical models of the field theory [7]. With regard to these considerations, we discuss some particular NSE solutions. However, in this discussion, the proposed approach is general and can successfully be applied to other types of effective potential.

We used the quadrature method to find solutions of the special form

$$\Psi(x,t) = \exp\{i\delta t + ipz\}y(z)$$

describing the propagation of nonspreading packets (solitary waves), where $z = x - Vt$ is the phase, $p = V/2$, and δ is a real number. In this approach, the solutions' behavior is analyzed by exploring the effective potential

$$U(y) = U(E_0, |y|) = E_0 y^2 - 2\int F(|y|)y\,dy, \quad E_0 = V^2/4 - \delta,$$

of the relevant ordinary differential equation

$$y_{zz} + E_0 y - F(|y|)y = 0.$$

Note that this method does not allow one to distinguish solitary waves from solitons which retain their structure when interacting with one another, because such a problem cannot be stated in the quadrature method. Instead, one should employ the method of inverse scattering problem [8, 9], which is not available for the NSE discussed therein. Therefore, for brevity, any solutions in the form of solitary waves obtained by the quadrature method will be referred to as solitons, kinks, oscillation or unstable NSE solutions.

Within this approach, the soliton-type solutions correspond to the motion of the imaging point between the local maximum of the effective potential (the stopping point) and the turning point. The kink-type solutions correspond to the imaging point's motion between two adjacent stopping points. The oscillation solutions correspond to the point's motion between two turning points. The unstable solutions are infinite, either limited by one singular point or having no singular points [2].

Let the equation $E - U(y) = 0$ have a solution for two real values, $0 \le y_1 < y_2$, and $U(y) < E$ in the variable domain $y_1 < y < y_2$. Let the effective potential $U(y)$ be presented by

$$U(y) = (|y| - y_1)^{\mu}(|y| - y_2)^{\nu} f(|y|) + E,$$

where $f(|y|)$ is a function (usually a polynomial) nonvanishing for $y_1 < y < y_2$.

Consider the integer values of μ and ν. At $\nu = 0$, the motion is infinite and corresponds to unstable NSE solutions. The values $\mu = \nu = 1$ correspond to oscillation solutions similar to mechanical motion between two turning points. A linearized (wavelike) NSE solution is valid in this case for the motion around the potential minimum.

The case $\mu = 2$, $\nu = 1$ (as well as $\mu = 1$ and $\nu = 2$) corresponds to an ordinary soliton NSE solution which decays exponentially at large phase times. For $1 \le \nu < 2$, the point y_2 is reached in a finite phase time, then the imaging point velocity changes its sign. That is why such points are referred to as turning points. For $\nu \ge 2$, it is impossible to reach the domain boundary y_2 in a finite phase time. Such points are referred to as stopping points.

At $\mu = 3$ and $\nu = 1$, the effective potential $U(y)$ has an inflection at $y = y_1$. In this case, the motion in the vicinity of the singular point $y = y_1$ occurs by a power law. Let us call the corresponding NSE solution a weak soliton. The asymptotic behavior of a weak soliton at large phase times (i.e., near the turning point) has the form $\Psi \sim (z - z_0)^{-2}$.

We consider an example of the separatrix solution to the NSE

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} - 2\beta|\Psi|\left(|\Psi| - \frac{3}{4}\right)\Psi, \quad \beta > 0,$$

for which the effective potential of the corresponding ordinary differential equation is $U(y) = \beta|y|^3(|y| - 1) + E_0 y^2$. Let us select the problem parameter E_0 such that $U(y)$ contains no quadratic term in y. The constant $E = 0$ corresponds to the separatrix. The quadrature yields an NSE solution in the form of two weak solitons (left-hand and right-hand) differing in their sign:

$$\Psi(x,t) = \mp\left(1 + \frac{\beta}{4}(z - z_0)^2\right)^{-1} \exp\{i(px - \mathcal{E}t)\}, \quad \mathcal{E} = p^2.$$

The turning points here correspond to the phase time $z - z_0 = 0$.

A similar power-function behavior is also inherent in other weak solitons for which $\mu > 2$ and $\nu = 1$,

$$\Psi \sim |z - z_0|^{2/(2-\mu)}.$$

An approximate shape of the potential $U(y)$ and the phase pattern for weak solitons are shown in Fig. 1.

It should specifically be pointed out that as $y \to 0$, weak solitons satisfy $y_z \sim |y|^{\mu/2}$ ($\mu > 2$), in contrast to ordinary solitons for which $y_z \sim |y|$ as $y \to 0$. At $\mu = \nu = 2$ the separatrix NSE solutions tend exponentially to y_1, y_2 (or to $-y_1$, $-y_2$) as the phase time increases. These are the right- and left-hand exponential kinks.

For $|\Psi| > y_2$, there is an unstable NSE solution on the separatrix $E = U(y_1) = U(y_2)$. At large phase times it behaves as $|\Psi| \sim |z - z_0|^{-1}$.

Note that if $U - E = 0$ only at one point $y_1 > 0$, then, by virtue of symmetry (i.e., evenness) of the potential $U(y)$, this equality also holds at the point $y = -y_1 < 0$. Then at $\mu = 1$ the points $y = \pm y_1$ are turning points, and the NSE solution has an oscillating character if $U(y) < E$ for $0 < |y| < y_1$.

For $\mu \geq 2$ ($\nu = 0$), there is an NSE solution in the form of a kink for which $|\Psi| \leq y_1$. In this case, the effective potential $U(y)$ can be presented as $U(y) = (|y| - y_1)^\mu f(y)$, where $U(y) < U(y_1)$ for $0 < |y| < y_1$. We notice that for an arbitrary effective potential the number of soliton solutions is even, while the number of kink solutions may be both even and odd.

At $\mu = 3$, $\nu = 2$, the NSE has a solution in the form of weak (power-function) right- and left-hand kinks, similarly to the case of $\mu = 3$, $\nu = 1$ for solitons. For $y \to y_1$, $y_z \sim |y - y_1|^{3/2}$.

An approximate shape of the potential and the phase pattern are shown in Fig. 2.

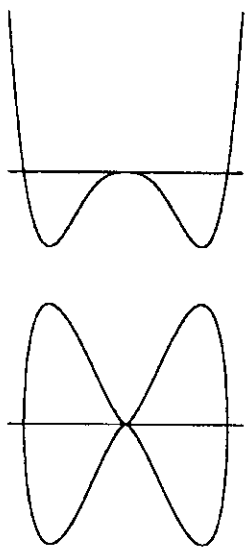

Fig. 1

Weak solitons.

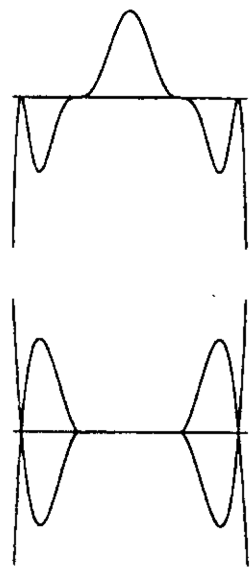

Fig. 2

Weak (power-function) kinks.

At $|\Psi| > y_2$, there is also an unstable separatix NSE solution with the behavior $|\Psi| \sim |y - y_1|^{-2/3}$ at large phase times.

At $\mu = 3$, $\nu = 0$, the NSE has a solution in the form of a single weak (power-function) kink. For example, for the potential $U(y) = \beta(y^2 - 1)^3$ the problem parameter (i.e., the coefficient at y_2) is $E_0 = 3\beta$, and the corresponding NSE has the form

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} - 3\beta|\Psi|^2(|\Psi|^2 - 2)\Psi.$$

On the separatrix (at $E = \beta$), the NSE has a solution in the form of a single weak kink

$$\Psi(x,t) = \frac{\beta^{1/2}(z - z_0)}{\sqrt{1 + \beta(z - z_0)^2}} \exp\{i(px - \mathcal{E}t)\}, \quad \mathcal{E} = E_0 + p^2.$$

The potential and phase pattern are shown in Fig. 3.

These examples confirm that the behavior of NSE solutions is determined by the nature of singular points, or, more precisely, by the value of the second derivative $U''(y)$ of the effective potential at these points. If $U''(y) = 0$ at a singular point, then this case corresponds to a weak (power-function) behavior of soliton or kink NSE solutions. However, the cases where $U''(y) \to -\infty$ at a singular point are possible. Then the imaging point will approach the singular point at high velocity, which corresponds to a faster-than-exponential decay of $|\Psi|$ at large phase times. Let us consider some examples in support of the abovesaid.

Consider a potential of the form $U(y) = U(|y|) = (E_0 - \beta)y^2 + 2\beta y^2 \ln|y|$ $(\beta > 0)$ which has a peculiar singularity at $y = 0$: $U(0) = 0$, $U'(0) = 0$, and the potential's second derivative behaves logarithmically at zero. This potential corresponds to the following NSE:

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} - 2\beta\Psi \ln|\Psi|.$$

In the vicinity of $y = 0$, $\dfrac{dy_z}{dy} \sim -\ln^{1/2}\dfrac{1}{|y|}$, which discerns this case from ordinary and weak solitons, where $\dfrac{dy_z}{dy} \sim |y|^{(\mu-2)/2}$ as $y \to 0$ $(\mu \geq 2)$. The exact NSE solutions on the separatrix $(E = 0)$ for an arbitrary value of E_0, which have the form [10]

$$\Psi(x,t) = \mp \exp\left\{\frac{1}{2}\left(1 - \frac{E_0}{\beta}\right)\right\} \exp\left\{-\frac{\beta}{2}(z - z_0)^2\right\} \exp\{i(px - \mathcal{E}t)\},$$

where $\mathcal{E} = E_0 + p^2$, will be called the (left- and right-hand) supersolitons. The potential and phase pattern are shown in Fig. 4.

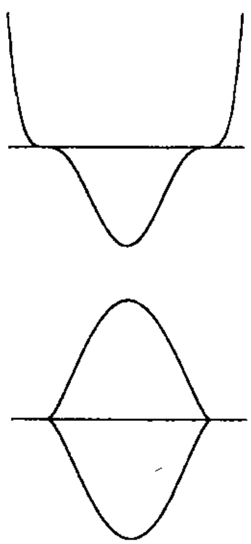

Fig. 3

Solitary weak kink.

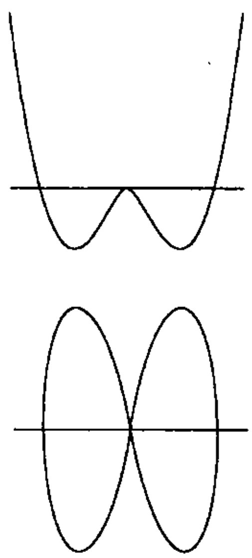

Fig. 4

Supersolitons.

Note that there exist the NSEs whose solutions decay faster than supersolitons at large phase times. In such cases, the second derivative of the effective potential at $y = 0$ must vary faster than $\ln|y|$ as $y \to 0$. For instance, if the effective potential $U(y)$ has the shape similar to that shown in Fig. 4, but in the vicinity of $y = 0$ it behaves as $U(y) \sim -y^2 \ln^{2-2/n}\left(|y|^{-1}\right)$, $n > 2$, then the separatrix $(E = 0)$ NSE solution at large phase times has the asymptotics

$$|\Psi| \sim \exp\left(-\gamma(z - z_0)^n\right), \quad \gamma > 0.$$

In conclusion, let us again draw the reader's attention to the terminology employed. Although we use, for brevity, the terms "soliton" and "kink" for solitary waves, all the weak (power-function) solutions and supersolitons discussed here are just solitary waves. This conclusion follows from the fact that the method of inverse scattering problem employed for finding soliton solutions to nonlinear equations leads automatically to exponential decay of nonreflective potentials at infinity [11].

We have demonstrated the possibility of the existence of new solution types for one-dimensional NSEs. Probably, these solutions may arise in description of domain walls in ferromagnet models or spin glasses. Then, it would be interesting to look for actual manifestations of these solutions in real physical experiments. Their mathematical structure allows for this, thus making the solutions very interesting from the mathematical standpoint and showing promise for observation in real nature.

REFERENCES

1. A.G. Kornienko, I.M. Ternov, A.R. Frenkin, and G.A. Chizhov, *Vestn. Mosk. Univ. Fiz. Astron.*, no. 4, p. 3, 1995.
2. A.G. Kornienko, A.R. Frenkin, and G.A. Chizhov, *Vestn. Mosk. Univ. Fiz. Astron.*, no. 5, p. 10, 1997.
3. V.G. Makhankov and V.K. Fedyanin, *Phys. Rep.*, vol. 104, no. 1, p. 1, 1984.
4. T.I. Belova and A.E. Kudryavtsev, *Usp. Fiz. Nauk*, vol. 167, no. 4, p. 377, 1997.
5. N.P. Erugin, *General Course in Differential Equations* (in Russian), Minsk, 1972.
6. A.I. Morozov and A.S. Sigov, *Usp. Fiz. Nauk*, vol. 164, no. 3, p. 243, 1994.
7. S.V. Ketov, *Nonlinear Sigma-Models in Quantum Field Theory and String Theory* (in Russian), Novosibirsk, 1992.
8. L.A. Takhtadzhyan and L.D. Faddeev, *Hamiltonian Approach in Soliton Theory* (in Russian), Moscow, 1986.
9. V.G. Makhankov, Yu.P. Rybakov, and V.I. Sanyuk, *Usp. Fiz. Nauk*, vol. 164, no. 2, p. 121, 1994.
10. A.G. Kornienko, *Graduate Thesis* (Moscow State University, Physics Faculty), Moscow, 1995.
11. V.B. Gostev and A.R. Frenkin, *Teor. Mat. Fiz.*, vol. 62, no. 3, p. 472, 1985.

16 June 1997 Department of Theoretical Physics