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DIMENSIONAL RENORMALIZATION WITHOUT NONINTEGER DIMENSIONS

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A renormalization scheme for generating functionals of Green functions is obtained. The suggested scheme is a version of the dimensional renormalization, but it uses finite integer positive dimensions only.

It is known that the presently popular renormalization scheme based on dimensional regularization [1] involves significant difficulties when applied to chiral and supersymmetric models. This is not surprising, because these models essentially use dimensional properties of the coordinate and momentum spaces. It is still unclear how to consistently formulate such properties in spaces formally having noninteger dimensions.

In this paper, we try to reformulate the dimensional renormalization, though, first, retaining its strong points such as a comparatively simple mathematical apparatus, and, second, eliminating non-integer dimensions.

The problem is considered within the renormalization scheme that we call the "renormalization along lines" [2]. The essence of the scheme is as follows. The Feynman amplitude $\mathcal{G}_s((p)_s^{\text{out}})$ that corresponds to a diagram G_s can formally be represented as

$$\mathcal{G}_s((p)_s^{\text{out}}) = \int dp\, D_l(p) * \mathcal{G}_{s/l}((p)_s^{\text{out}}, p, -p). \tag{1}$$

Here $\mathcal{G}_{s/l}$ is the Feynman amplitude corresponding to the diagram $G_{s/l}$ that is obtained from the diagram G_s by breaking the inner line l. $D_l(p)$ is the propagator corresponding to this line, $*$ denotes the convolution of this propagator with respect to indices omitted in (1) and the amplitudes $\mathcal{G}_{s/l}$; $(p)_s^{\text{out}}$ is the set of all external momenta of the diagram G_s; and $(p)_s^{\text{out}}, p, -p$ is the same set for the diagram $G_{s/l}$.

Unfortunately, (1) does not always have a clear mathematical meaning, because the integral in the right-hand side may diverge due to ultraviolet divergences.

In [2], multiplication by the propagator $D_l(p)$, convolution with respect to its indices, and integration with respect to momentum p are considered as a formal definition of the unique linear operator

$$\int dp\, D_l(p) * \ldots \tag{2}$$

which acts on the $\mathcal{G}_{s/l}$ amplitude. If the integral in the right-hand side of (1) diverges, operator (2) can be replaced by its linear extension which is somewhat conventionally written as

$$\int d\mu(p)\, D_l(p) \ldots . \tag{3}$$

The conventionality arises because the "renormalized integration" $\int d\mu(p) \ldots$ introduced in (3) must include a certain subtraction procedure, and we cannot require that the measure μ be positive definite.

It is shown in [2] that if such an extension exists, then the formula

$$\mathcal{G}_s((p)_s^{\text{out}}) = |L_s|^{-1} \int d\mu(p)\, D_l(p) \sum_{l\in G_s} \mathcal{G}_{s/l}((p)_s^{\text{out}}, p, -p) \tag{4}$$

recursively specifies the amplitude $\mathcal{G}_s((p)_s^{\text{out}})$, and this amplitude coincides with the renormalized Feynman amplitude constructed by applying the Bogolyubov–Parasyuk R-operation [3] with a certain fixation in the final renormalization.

Formula (4) is written under the assumption that all lines (propagators) in the model are distributed over a finite number of ordered types: the first, the second, ..., the N-th type. Partition according to types can be arbitrary but the same for all diagrams and subdiagrams. For example, the first type may include the propagators of all fermion fields, the second one, of electromagnetic fields, and the third one, of all other fields. The summation in the right-hand side of (4) is carried out over all lines l of a certain type which are inner for the diagram G_s. It is assumed that the amplitudes $\mathcal{G}_{s/l}$ are obtained similarly, and the summation is performed over the lines of similar type if they exist in the $G_{s/l}$ diagram; otherwise, the summation is performed over the preceding-type lines.

Let us now obtain an analogue of formula (4) for Green functions, or, to be more precise, for generating functionals $Z(j)$ of Green functions. In what follows, it is more convenient to reason assuming that the space of momenta is endowed with the Euclidean metric.

Let the system dynamics be described by the Euclidean action

$$S_E(\varphi) = \sum_{n=1}^{N} I_E^n(\varphi) + W_E(\varphi) \tag{5}$$

which includes both physical and wind fields contributions and, if necessary, the terms that fix gauge. The set φ of all fields is partitioned into N types,

$$\varphi = \{\varphi_u^n\}, \qquad n = 1, \ldots, N.$$

Here, the subscript u labels the fields of one type, which may be different or conjugate. Each term $I_E^n(\varphi)$ in the right-hand side of (5) depends only on the type-n fields, is quadratic with respect to these fields, and, under quantization, gives the relevant Euclidean propagator,

$$\langle \varphi_u^n(p)\varphi_v^n(p')\rangle_E = iD_{uv}^n(p)_E\,\delta(p+p').$$

As is known, the generating functional can formally be represented as

$$Z(j) = \mathcal{N}^{-1} \exp\left\{-W\left(\frac{\delta}{\delta j}\right)\right\} \exp\left\{\frac{i}{2}\int dp_E \sum_{n=1}^{N}\sum_{u,v} j_u^n(p) D_{uv}^n(p)_E\, j_v^n(-p)\right\}, \tag{6}$$

where $\mathcal{N}$ is the normalizing factor and $j_u^n(p)$ are the flows; variation of the $Z(j)$ functional with respect to these flows gives the corresponding Green functions. We multiply the right-hand side of (6) by $C_E(j,\varphi)|_{\varphi=0}$, where

$$C_E(j,\varphi) = \exp\left\{\sum_{n=1}^{N}\int dp_E \sum_u j_u^n(p)\varphi_u^n(p)\right\},$$

perform variation with respect to j, and finally replace j by variation with respect to φ. As a result, (6) takes the form

$$Z(j) = \mathcal{N}^{-1}\prod_{n=1}^{N} \exp\left\{\frac{1}{2}\int dp_E \sum_{u,v} \frac{\delta}{\delta\varphi_u^n(p)} D_{uv}^n(p)_E \frac{\delta}{\delta\varphi_v^n(-p)}\right\} \exp\{-W_E(\varphi)\} C_E(j,\varphi)\Big|_{\varphi=0}. \tag{7}$$

Of course, (7) cannot be applied to finding Green functions in practice because the integrals with respect to p in (7) may diverge. However, it is easy to guess how this formula should be corrected. The linear operator

$$\int dp_E \sum_{u,v} \frac{\delta}{\delta\varphi_u^n(p)} D_{uv}^n(p)_E \frac{\delta}{\delta\varphi_v^n(-p)} \tag{8}$$

formally adds one additional inner line of type n to each Green function. For Feynman amplitudes, this is made with the use of operator (2) by formula (1). To remove divergences in (1), we should replace operator (2) by its linear extension (3). Similarly, to remove divergences in (7), we should replace operator (8) by its linear extension

$$\Delta_E^n = \int d\mu(p) \sum_{u,v} \frac{\delta}{\delta\varphi_u^n(p)} D_{uv}^n(p)_E \frac{\delta}{\delta\varphi_v^n(-p)}. \tag{9}$$

Formula (7) then becomes

$$Z(j) = \mathcal{N}^{-1} \exp\{\Delta^N/2\} \ldots \exp\{\Delta^1/2\} \exp\{-W_E(\varphi)\} C_E(j,\varphi)|_{\varphi=0}. \tag{10}$$

Formula (10) is an analogue of (4). However, (10) is not recursive and does not feature forced symmetrization with respect to inner lines since symmetrization is automatically performed.

The generating functional for renormalized Green functions can actually be obtained with the use of (10) by developing the operation $\int d\mu(p)$ of "renormalized integration" included in (9) (and in (4)). To this end, we first apply the procedure described in the so-called "renormalization with respect to asymptotics". Its initial version was suggested in [4] where the space of momenta was assumed four-dimensional. Later, it was however found that such a procedure became ambiguous when applied to diagrams with a sufficiently large number of external momenta. This shortcoming was removed in [5] where the second, improved version of the procedure was described. The improvement was made at a fairly high cost of using, i.e., momentum spaces of dimensions other than four at intermediate stages. This manifests explicitly an analogy to dimensional regularization. However, unlike dimensional regularization, regularization with respect to asymptotics involves integer dimensions only, which may be made all even. For diagrams of finite orders with respect to the coupling constant, finite dimensions are sufficient.

The procedure for constructing the operation of renormalized integration is based on the theorem proved in [5].

Suppose that (i) p, k_i, and k_j, where $i,j = 1,\ldots,r$, are the vectors in the 2ζ-dimensional ($\zeta \geq 2$) space; (ii) in the space of dimension d ($r \leq d < 2\zeta$), all scalars p^2, pk_i, and k_ik_j are independent; and (iii) $F(p^2, pk_i, k_ik_j)$ is a scalar function of scalar arguments and is absolutely integrable over a compact domain and can be presented as

$$F(p^2, pk_i, k_ik_j) = F_1(p^2, pk_i, k_ik_j) + F_2(p^2, pk_i, k_ik_j), \tag{11}$$

where F_1 is absolutely integrable in the compact domain; $(p^2)^2F_1$ decreases as $|p| \to \infty$, and $(p^2)^2F_2$ is a polynomial in p^2, pk_i, and $\ln p^2$. Then in the formula

$$\Phi(k_ik_j,\varepsilon) = (-1)^\zeta(\mu^2)^{-\varepsilon}\pi^{2-\zeta+\varepsilon}\Gamma^{-1}(\varepsilon) \int d^{2\zeta}p \int_0^\infty d\omega^2(\omega^2)^{\varepsilon-1} \left(\frac{\partial}{\partial\omega^2}\right)^{\zeta-2} U(\omega)\,\mathcal{P}\,F(p^2, pk_i, k_ik_j) \tag{12}$$

all integrals converge absolutely; Φ does not depend on ζ if $2\zeta > d$ and is a scalar function of independent scalar arguments k_ik_j and an analytic function of ε in the domain $0 \leq \operatorname{Re}\varepsilon < 1$ with possible poles at $\varepsilon = 0$; Γ is the gamma function; μ is the mass parameter; and $\mathcal{P}$ and U are the operators defined by

$$\mathcal{P}\,F = F_1, \qquad U\,F_1(p^2, pk_i, k_ik_j) = F_1(p^2 + \omega^2, pk_i, k_ik_j).$$

In what follows, we restrict our consideration to models with scalar fields. Vector and spinor fields will be considered elsewhere. The Weinberg theorem [6] and its generalization [7] imply that, in scalar models, the renormalized Green functions and the Feynman amplitudes multiplied by the propagator $D_l(p)$ are either functions of type (11) or functions (11) multiplied by delta functions of linear combinations of momenta p and k_i. The case where the delta-function argument contains the integration momentum p is trivial, for the integral with respect to p then converges and operators (2) and (8) should not be extended. In the rest cases, renormalized integration can be defined by the formula

$$\int d\mu(p) \ldots = \mathcal{L}(\varepsilon \downarrow 0)(-1)^\zeta(\mu^2)^{-\varepsilon}\pi^{2-\zeta+\varepsilon}\Gamma^{-1}(\varepsilon) \int d^{2\zeta}p \int_0^\infty d\omega^2\,(\omega^2)^{\varepsilon-1} \left(\frac{\partial}{\partial\omega^2}\right)^{\zeta-2} U(\omega)\,\mathcal{P} \ldots \tag{13}$$

Here $\mathcal{L}(\varepsilon \downarrow 0)$ is the operation of Laurent series expansion in ε with preservation of a single term of the order ε^0.

Formula (13) closely resembles the integration operation in the space of noninteger dimension $2\zeta + 2\varepsilon$. However, (13) possesses two advantages. First, the space of momenta p has the integer dimension 2ζ which facilitates substantially an analysis of the Green functions symmetry properties where the space dimension is essential. Second, the integration final result does not depend on ζ for sufficiently large ζ since, e.g., in (13), the 2ζ-dimensional integration with respect to p and the $(\zeta - 2)$-fold differentiation with respect to ω^2 partially compensate each other.

At the same time, (13) differs unfavorably from the similar integration formula in spaces of noninteger dimensions in that it contains the $\mathcal{P}$ operator. For simple diagrams, it is fairly easy to decompose the integrand according to (11) and to explicitly construct the $\mathcal{P}$ operator, though for complex diagrams, this is rather difficult.

For this reason, it is convenient to slightly transform (13) as follows:

$$\int d\mu(p)\, F(p^2, pk_i) = \mathcal{L}(\varepsilon \downarrow 0)(-1)^{\zeta-2}(\mu^2)^{-\varepsilon}\pi^{2-\zeta+\varepsilon}\Gamma^{-1}(\varepsilon) \times \lim_{\beta\to 0}\int d^{2\zeta}p\,(p^2+\mu^2)^{-\beta}\int_0^\infty d\omega^2\,(\omega^2)^{\varepsilon-1}\left(\frac{\partial}{\partial\omega^2}\right)^{\zeta-2} F(p^2+\omega^2, pk_i). \tag{14}$$

Here, the $\beta \to 0$ limit acts as an analytic continuation to zero from the values of β at which the integrals in ω and p converge. In (14), it is assumed that the function $F(p^2, pk_i)$ admits expansion (11); its arguments k_ik_j are omitted as inessential for the following consideration.

We note that the operators $U(\omega)$ and $\mathcal{P}$ are absent in (14), though the $U(\omega)$ operation is explicitly performed. The $\mathcal{P}$ operation is also present in (14) yet implicitly. To verify this statement we consider F as a function of vectors p and k_i ($i = 1, \dots, r$) in a space $S_{2\zeta}$ of dimension 2ζ. Suppose that the vectors k_i lie in a subspace S_d of this space. Denote the projections of p onto the unit vectors of the space S_d by $p_1, \dots, p_d$ and consider integral (14) of the functions F of the form

$$C(k_i)(p_1)^{2\alpha_1}\dots(p_d)^{2\alpha_d}(p^2)^{\alpha}(\ln p^2)^{\alpha_0}.$$

Here, $\alpha_0, \alpha_1, \dots, \alpha_d$ are nonnegative integers and $\alpha \geq -\tilde{\alpha}$, where $\tilde{\alpha} = 2 + \alpha_1 + \dots + \alpha_d$. At $\alpha_0 = 0$, integral (14) is explicitly evaluated and gives an expression of the form

$$\mathcal{L}(\varepsilon \downarrow 0)\lim_{\beta\to 0}(\mu^2)^{-\beta}\Gamma^{-1}(\beta)\Gamma(\beta - \alpha - \tilde{\alpha} - \varepsilon)\mathcal{M}, \tag{15}$$

where $\mathcal{M}$ includes all inessential factors, finite and independent of β. Tending to the $\beta \to 0$ limit, (15) vanishes. At $\alpha_0 \neq 0$, integral (14) is evaluated by the α_0-fold differentiation of (15) with respect to α. As a result, we again obtain zero.

This implies that, for functions F of form (11), the right-hand side of (14) does not change under replacement of F by $\mathcal{P}F$; therefore, we can assume that the $\mathcal{P}$ operator in the right-hand side of (14) is effectively present.

We found that the result of application of (14) coincides with that obtained in the formalism of dimensional regularization in the version suggested by Wilson [8] (see also [9]). This can be demonstrated as follows. Since in the right-hand side of (14) F is subjected to differentiation, subtraction of first several terms of the Maclaurin expansion from F and integration by parts will not affect the result. Then the expression in the right-hand side of (14) after the operator $\mathcal{L}(\varepsilon \downarrow 0)$ will be rewritten as

$$(\mu^2)^{-\varepsilon}I(\varepsilon) \equiv (\mu^2)^{-\varepsilon}\lim_{\beta\to 0}\int d^{2\zeta}p\,(p^2+\mu^2)^{-\beta}I_\perp(p;\varepsilon-\zeta+2), \tag{16}$$

where

$$I_\perp = \pi^{2-\zeta+\varepsilon}\Gamma^{-1}(\varepsilon-\zeta+2)\int_0^\infty d\omega^2\,(\omega^2)^{\varepsilon-\zeta+1}\left[F(p^2+\omega^2, pk_i) - \sum_{l=0}^{\zeta-3}\frac{(\omega^2)^l}{l!}F^{(l)}(p^2, pk_i)\right]. \tag{17}$$

In Wilson's terminology, (17) is the definition of the integral of the function $F(p^2 + \omega^2, pk_i)$ over the "transverse space" of noninteger dimension $2(\varepsilon - \zeta + 2)$, and (16) defines $I(\varepsilon)$ as the integral of $I_\perp(p)$ over the "longitudinal space" of positive integer dimension 2ζ. Equations (16) and (17) define jointly $I(\varepsilon)$ as the integral of $F(p, k_i)$ of noninteger dimension $(4+2\varepsilon)$. If desired, the well-developed technique for integration over spaces of noninteger dimensions can be used. It should only be borne in mind that each integration is performed at the specified ε value (ε_1, ε_2, etc.) Passage to physical dimensions requires sequentially applying the $\mathcal{L}(\varepsilon_2 \downarrow 0)$, $\mathcal{L}(\varepsilon_1 \downarrow 0)$, etc. operations to the integration results.

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REFERENCES

1. G. v'tHooft and M. Veltman, *Nucl. Phys.*, vol. B44, p. 189, 1972.
2. D.A. Slavnov, *Teor. Mat. Fiz.*, vol. 110, p. 399, 1997.
3. N.N. Bogolyubov and O.S. Parasyuk, *Izv. Akad. Nauk SSSR. Ser. Mat.*, vol. 20, p. 585, 1956.
4. V.A. Il'in, M.S. Imashev, and D.A. Slavnov, *Teor. Mat. Fiz.*, vol. 52, p. 177, 1982.
5. D.A. Slavnov, *Teor. Mat. Fiz.*, vol. 62, p. 335, 1985.
6. S. Weinberg, *Phys. Rev.*, vol. 118, p. 838, 1960.
7. D.A. Slavnov, *Teor. Mat. Fiz.*, vol. 17, p. 169, 1973.
8. K.G. Wilson, *Phys. Rev.*, vol. D7, p. 2911, 1973.
9. J.C. Collins, *Renormalization*, Cambridge Univ. Press, Cambridge, 1984.

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