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FINSLERIAN INVARIANT AND COORDINATE LENGTHS IN THE INERTIAL REFERENCE FRAME

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Basing on Finslerian invariances, temporal and spatial scales in inertial reference frames are defined. The respective Finslerian metric function is indicated, and the Finslerian deformation coefficients are calculated.

Continuing our works [1–2] and using the notation adopted therein, we consider two reference frames S_0 and $S(v)$, applying the Finslerian kinematic transformations

$$T = H^0_{(0)}t + H^0_{(1)}x,$$
$$X = H^1_{(0)}t + H^1_{(1)}x, \quad Y = H^2_{(2)}y, \quad Z = H^3_{(3)}z \tag{1}$$

and their vector representation

$$X^P = H^P_{(Q)}x^Q, \tag{2}$$

where $X^P = (T, X, Y, Z)$ refers to S_0 and $x^P = (t, x, y, z)$ refers to $S(v)$; $P, Q, R = 0, 1, 2, 3$. The tetrad representation

$$g_{PQ}(v) = \sum_{R=0}^{3} q_R H^{(R)}_P(v) H^{(R)}_Q(v) \tag{3}$$

is valid for the Finslerian metric tensor, where

$$q_P = (1, -1, -1, -1) \tag{4}$$

in accordance with the space-time signature of the tensor g_{PQ}. The explicit form of the components $[H^P_{(Q)}(v), H^{(Q)}_P(v)]$ is given in [1]. We shall restrict ourselves to the case $j = 1$.

Kinematic transformations (1)–(2) have a pure-passive meaning, for they specify the variation rules for the vector components in going from S_0 into $S(v)$. Deformation of the proper scales (time and length primary standards) in the reference frame $S(v)$ due to its motion relative to the reference frame S_0 is a profound physical reason of these variations. This also implies that the four-dimensional vectors $[X^P]$ themselves remain unchanged, keeping their directions in the four-dimensional space-time.

As in [1, 2], we start by stipulating that the x^1- and X^1-axes of the reference frames S_0 and $S(v)$ are parallel and that the components v^2 and v^3 of the three-dimensional velocity vector $\mathbf{v}$ vanish. We shall adopt the notation $v = v^1$, so that $S(v)$ moves along or opposite to the x^1-axis for $v > 0$ or $v < 0$, respectively.

The definition

$$||X||_{S(v)} \overset{\text{def}}{=} g_{PQ}(v) X^P X^Q \tag{5}$$

specifies the X^P vector coordinate length with respect to the reference frame $S(v)$. Using transformations (2)–(4), we obtain

$$||X||_{S(v)} = (x^0)^2 - |\mathbf{x}|^2, \quad |\mathbf{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{6}$$

Thus the following is valid.

Proposition 1. *In $S(v)$ the coordinate length of four-dimensional vectors is defined by ordinary pseudo-Euclidean rule* (6) *independent of Finslerian extension.*

In fact, this statement is a direct corollary of space-time signature (4) of the Finslerian metric tensor. It can also be said that the observation space in the reference frame $S(v)$ is pseudo-Euclidean. Since $g_{PQ}(v=0) = q_P\delta_{PQ}$ (an implication of equalities (3) and (4)), definition (6) coincides with the definition (5) used in the primary preferred reference frame S_0. This coincidence does not exist in the reference frame $S(v)$ under the Finslerian treatment.

At the same time, the $S(v)$-coordinate length (5) is not invariant under transformations (1)–(2) which are more general than the Lorentzian transformations.

Now, let a Finslerian metric function $F(T, \mathbf{X})$ exist, invariant under transformations (1)–(2), so that

$$F(T, \mathbf{X}) = F(t, \mathbf{x}). \tag{7}$$

We call

$$||X||_F \stackrel{\text{def}}{=} F(X^0, \mathbf{X}) \tag{8}$$

the F-length of the four-dimensional vector (X^P). According to (7), it is invariant under kinematic transformations (1)–(2). The latter invariance property makes it possible to use the F-length to coordinate and gauge the scales for time and space standards in different reference frames.

Let $(\widetilde{X}^P)$ and $(\widetilde{\widetilde{X}}^P)$ be two time-like vectors. They have, respectively, the components $(\widetilde{x}^0, 0, 0, 0)$ and $(\widetilde{\widetilde{x}}^0, 0, 0, 0)$ in their proper reference frames. As to the physical sense of components $\widetilde{x}^0$ and $\widetilde{\widetilde{x}}^0$, they mean the proper time intervals. Since

$$F(x^0, 0, 0, 0) \equiv x^0 \tag{9}$$

(owing to invariance (7)), the equalities

$$\widetilde{x}^0 = F(\widetilde{X}^P), \quad \widetilde{\widetilde{x}}^0 = F(\widetilde{\widetilde{X}}^P) \tag{10}$$

hold true.

Thus we see that the following is valid.

Proposition 2. *The Finslerian F-length of a time-like vector has a clear meaning: the length is equal to the proper time interval corresponding to the vector. In particular, as a consequence of* (10), *the equality of proper time intervals, $\widetilde{x}^0 = \widetilde{\widetilde{x}}^0$, holds if and only if the corresponding F-lengths are equal, i.e., if*

$$F(\widetilde{X}^P) = F(\widetilde{\widetilde{X}}^P). \tag{11}$$

The proper time is determined in terms of the Finslerian F-length (which differs from the $S(v)$-coordinate length). The identity

$$F(T, 0) = T \tag{12}$$

is valid.

Problem. *What invariant meaning should be assigned to the concept of "the line segments of equal length in the reference frame $S(v)$"?*

Let us consider in $S(v)$ an intercept of x^1-axis and denote its length by x. Geometrically, the intercept is represented by the four-dimensional vector $x^p = (0,\ x,\ 0,\ 0)$. Transforming this vector from $S(v)$ to S_0 in accordance with rule (1), we get

$$T = xv/V(v), \quad X = x(1 - g|v|)/V(v) \tag{13}$$

(formulas (59) from [1] are used). If F is the Finslerian metric function relating to this case, then the following equality should be valid:

$$F(T, X) = xF(v,\ 1 - g|v|)/V(v). \tag{14}$$

In its physical sense, the concept of invariant spatial length implies the relation

$$x = F(T, X) \tag{15}$$

(cf. (10)). In view of (15), relation (14) holds if and only if the function F obeys the identity

$$F(v,\, 1 - g|v|) = V(v). \tag{16}$$

The best agreement with conventional notions in S_0 is achieved when the stipulations can be supplemented by the identity

$$F(0, \mathbf{X}) = |\mathbf{X}|, \quad |\mathbf{X}| = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2} \tag{17}$$

($F > 0$ is meant). Conditions (17) and (12) are tantamount.

In [1], invariance condition (7) has been used to derive the Finslerian metric function $F(T, \mathbf{X})$ for the time-like vectors (X^p), where $|\mathbf{X}|/|T| < g_+$. It is this function that should be used in (8)–(12). The derivation procedure of [1] is applicable to the case of space-like vectors (X^P), i.e., for $|\mathbf{X}|/|T| > g_+$. Omitting the expositions, we write the eventual result obtained,

$$F(T, \mathbf{X}) = (|\mathbf{X}| - g_+|T|)^{g_+/2A}(|\mathbf{X}| - g_-|T|)^{-g_-/2A}, \qquad |\mathbf{X}|/|T| > g_+. \tag{18}$$

Assuming

$$F(T, \mathbf{X}) = |\mathbf{X}| Z(q), \quad q = |T|/|\mathbf{X}| < 1/g_+, \tag{19}$$

we find the respective generating function

$$Z(q) = (1 - g_+ q)^{g_+/2A}(1 - g_- q)^{-g_-/2A}. \tag{20}$$

Since $Z(0) = 1$, it follows from (19) that we can pose

Proposition 3. *Identity* (17) *is fulfilled under choice* (18).

Does function (18) obey identity (14)? We verify: $1 - g|v| - g_\pm|v| = 1 - (g_\pm + g)|v| = 1 + g_\mp|v|$ (formula (36) of [1] is used). Therefore, the left-hand part in (14) should be equal to the function $(1 + g_-|v|)^{g_+/2A}(1 + g_+|v|)^{-g_-/2A}$. The latter function, however, is precisely $V(v)$ (see (34) in [1]).

Thus we proved the validity of

Proposition 4. *With Finslerian metric function* (18), *identity* (14) *holds over the range* $|v| < g_+$.

Proposition 5. *Finslerian metric function* (18) *defines the invariant space length in accordance with* (15).

If $(\widetilde{X}^p)$ and $(\widetilde{\widetilde{X}}^P)$ are two vectors in the $(q < g_+)$-region, then the corresponding proper space-like lengths are equal if and only if $F(\widetilde{X}^P) = F(\widetilde{\widetilde{X}}^P)$, i.e., the condition of the form (11) is applicable to this case. Thus we have given a complete solution to the problem formulated above.

It is also interesting to consider in $S(v)$ a vector $x^P = (0, 0, y, 0)$, that is, a length element y on the x^2-axis in $S(v)$. Using (1), we get $Y = y/d(v)$. ($d(v)$ is the function found in [1] and specified by formula (39) of [1].) According to (17), $F(0, Y) = Y$. Whence we proved

Proposition 6. *If the direction of a segment is perpendicular to the three-dimensional vector* $\mathbf{v}$, *then the* F*-invariant length of the segment is not deformed.*

Proposition 7. *The* $S(v)$*-coordinate lengths* x *and* y *are equal in the* F*-invariant sense if and only if*

$$Y = d(v)x. \tag{21}$$

Conversely, the $S(v)$-invariant lengths x and y are equal in the reference frame $S(v)$ at $x = y$, while the $S(v)$-coordinate length y suffers deformations (in accordance with the law $y = d(v)Y$, where $d(v) \neq 1$ in the Finslerian approach) as a result of $S(v)$ reference frame motion with respect to the S_0 reference frame. Thus Propositions 6 and 7 should be taken into account under thorough analysis of relevant relativistic experiments.

Now we introduce the following

Definition. *Let L be a three-dimensional surface in the reference frame $S(v)$ formed by the ends of segments issuing from the origin (0,0,0,0) of $S(v)$. If each point $(x, y, z) \in L$ relates to a fixed F-invariant length R, then L is called the surface of constant F-invariant length R in $S(v)$ or, for brevity, the R-surface in $S(v)$.*

The nonzero Finslerian parameter g leads to the effect that the R-surface ceases to be an (ordinary) sphere!

To obtain the equation of the R-surface, we have to consider in $S(v)$ the space-like vectors $x^P = (0, x, y, z)$, to perform their transformation in S_0 in accordance with (1), and then to calculate $q = |T|/|\mathbf{X}|$. This yields

$$q = q(g; v, \theta) = |v\cos\theta|/[Q(v) + (v^2 - g|v| + g^2v^2)\cos^2\theta]^{1/2}. \tag{22}$$

We use the notation

$$r = [x^2 + y^2 + z^2]^{1/2}, \quad x = r\cos\theta. \tag{23}$$

Substituting (22) into (20), we find from (19)

Proposition 8. *The equation of the R-surface in $S(v)$ has the form $r = r(\theta)$, where*

$$r(\theta) = RV(v)/[(1 - g|v|)Z(q)\cos\theta]. \tag{24}$$

For $q \ll 1$, function (20) can be expanded as follows:

$$Z(q) = 1 + gq - \frac{1}{2}q^2 + \frac{1}{6}gq^3 - \frac{1}{8}\left(1 + \frac{2}{3}g^2\right)q^4 + O(5), \tag{25}$$

$$Z^2(q) = 1 + 2gq - q^2 - \frac{2}{3}gq^3 + \frac{1}{6}g^2q^4 + O(5), \tag{25a}$$

so that accounting for only the lowest corrections in g in (19) yields the approximate expression

$$F^2(T, \mathbf{X}) = \mathbf{X}^2 - T^2 + 2g|\mathbf{X}||T|, \tag{26}$$

which is the nearest Finslerian approximation for the ordinary Lorentzian length definition $|\mathbf{X}|^2 - T^2$ of space-like vectors. Substituting (1) into (26) and using (23), we get the representation

$$F^2(T, \mathbf{X}) = r^2(\theta)[1 - g|v|(1 - |\cos\theta|)^2], \tag{27}$$

which implies that in this lowest approximation function (24) can be taken in the simple form

$$r(\theta) = n(\theta)R, \quad n(\theta) = 1 + \frac{1}{2}g|v|(1 - |\cos\theta|)^2. \tag{28}$$

From (28) it follows directly that

$$r(0) = R, \quad n(0) = 1, \tag{29}$$

$$r(\theta) = r(\theta + \pi), \tag{30}$$

whence the function

$$u(\theta) \stackrel{\text{def}}{=} r\left(\theta + \frac{\pi}{2}\right)/r(\theta) \tag{31}$$

is

$$u(\theta) = 1 + \frac{1}{2}g|v|[2(|\cos\theta| - |\sin\theta|) - \cos^2\theta + \sin^2\theta)]. \tag{32}$$

Subject to the expression

$$c(\theta) = 1 + \frac{1}{2}g|v|(1 + \cos^2\theta) \tag{33}$$

for the light signal velocity (formula (49) in [2]), relation (28) entails

$$n(\theta)/c(\theta) = 1 - g|v||\cos\theta|, \tag{34}$$

which reduces the function

$$p(\theta) \stackrel{\text{def}}{=} [c(\theta)]^{-1} - u(\theta)\left[c\left(\theta + \frac{\pi}{2}\right)\right]^{-1} \tag{35}$$

to

$$p(\theta) = g|v|(|\sin\theta| - |\cos\theta|) \tag{36}$$

(relation (32) was used). The equality

$$p(\theta) = -p\left(\theta + \frac{\pi}{2}\right) \tag{37}$$

holds true together with

$$c(\theta + \pi) = c(\theta + 2\pi) = c(\theta), \quad c\left(\theta + \frac{3\pi}{2}\right) = c\left(\theta + \frac{\pi}{2}\right). \tag{38}$$

Now function (22) can be rewritten as

$$q = |v\cos\theta|c(\theta). \tag{39}$$

Equality (30) points to "the mutual equivalence of opposite directions in the reference frame $S(v)$", so we have

Proposition 9. *Turning a segment by an angle 180° does not change the segment F-length.*

Relation (28) also implies that $n(\theta)$ simulates "the deformation coefficient of an x^1-axis intercept as it turns by an angle θ". In other words, (28) and (29) lead to the validity of the following

Proposition 10. *Given an intercept of the x^1-axis. If x is its length, then turning the intercept by an angle θ should change the F-length to*

$$s(\theta) = n(\theta)x. \tag{40}$$

The meaning of function (31) is also evident: the function presents the deformation value of the intercept under its turn by an angle 90°. Thus we have proven

Proposition 11. *If a segment in $S(v)$ makes an angle θ with the x^1-axis and has the length $l(\theta)$, then its turn by 90° changes its F-length to $l(\theta + \pi/2)$ in accordance with the equality*

$$l(\theta + \pi/2) = u(\theta)l(\theta). \tag{41}$$

We use the above formulas to consider the Michelson–Morley-type experiment. Let there be given in $S(v)$ two mutually perpendicular segments of lengths l_1 and l_2 arranged at angles θ and $\theta + \pi/2$ to the x^1-axis, respectively, and issuing from the center point $C = (0, 0, 0, 0)$ of $S(v)$. Two light signals emerge from the point C, one signal follows along the l_1-segment, reflects back at the end of the segment, and returns to the departure point C, while the other signal goes similar way along the segment l_2. Clocks located at the point C show a value τ for the difference between the respective times of travel. Thereupon the $l_1 l_2$-set is turned through 90° and the procedure is repeated yielding some new value τ'. We have

$$\tau = \frac{l_1}{c(\theta)} + \frac{l_1}{c(\theta + \pi)} - \frac{l_2}{c(\theta + \pi/2)} - \frac{l_2}{c(\theta + 3\pi/2)},$$

$$\tau' = \frac{l_1 u(\theta)}{c(\theta + \pi/2)} + \frac{l_1 u(\theta)}{c(\theta + 3\pi/2)} - \frac{l_2 u(\theta + \pi/2)}{c(\theta + \pi)} - \frac{l_2 u(\theta + \pi/2)}{c(\theta + 2\pi)}.$$

Applying formulas (35)–(38) yields the following simple relation:

$$\frac{1}{2}(\tau - \tau') = (l_1 + l_2)p(\theta), \quad p(\theta) = g|v|(|\sin\theta| - |\cos\theta|). \tag{42}$$

In particular,

$$\frac{1}{2}(\tau - \tau') = \begin{cases} -(l_1 + l_2)g|v| & \text{at} \quad \theta = 0, \\ 0 & \text{at} \quad \theta = 45°, \\ (l_1 + l_2)g|v| & \text{at} \quad \theta = 90°. \end{cases} \tag{43}$$

Thus the Finslerian effect features essential anisotropy, as shown by derived formulas (42)–(43).

The Finslerian function $Z(q)$ (Eq. (20)) refers to the case of space-like vectors X^P. Its counterpart for the case of time-like vectors X^P is the function $V(v)$ (given by Eq. (34) in [1]). In the Riemannian (pseudo-Euclidean) limit, we have $Z(q)|_{g=0} = \sqrt{1-(T/|\mathbf{X}|)^2}$ and $V(v)|_{g=0} = \sqrt{1-(|\mathbf{X}|/T)^2}$.

The functions $Z(q)$ and $V(v)$ differ radically in the ways of expansion in their variables $q \ll 1$ and $v \ll 1$. Namely, the Finslerian corrections appear in the expansion of $V(v)$ in v beginning only with the $O(gv^3)$ terms, i.e., those of rather high order of smallness. We observe, however, that subject to (25), the lowest-order corrections $O(gq)$ really contribute to the $Z(q)$ expansion in g! This is the latter circumstance that led to the $O(gv)$-corrections in (26)-(36). The presence of such corrections and a sharp contrast in the approximation behavior of $Z(q)$ and $V(v)$ are the phenomena which could hardly be foreseen before the explicit form of $Z(q)$ (20) had been found. The only common feature of considered expansions is that $Z(q)$ does not involve corrections of the type $O(gq^2)$ and $V(v)$ corrections of the type $O(gv^2)$.

To summarize: Propositions $1-11$ proven above answer the question "What the Finslerian deformations of time- and space-standards should mean?" and simultaneously put forward succinct and sufficient analytic tools to compute the deformations. In particular, Propositions 9–11 indicate the existence of a whole host of $O(gv)$-level corrections which should be accounted for in the course of thorough Finslerian analysis of the known relativistic experiments (see, e.g., [3, 4]), or in forecasting new post-Lorentzian relativistic experiments. For example, simple Finslerian $O(gv)$-corrections $(42)-(43)$ to the outcomes of Michelson–Morley-type experiments appear. Having explicit functions $Z(q)$ and $V(v)$ opens up straightforward ways for computing corrections of any order of smallness.

REFERENCES

1. G.S. Asanov, *Vestn. Mosk. Univ. Fiz. Astron.*, no. 1, p. 18, 1996.
2. G.S. Asanov, *Vestn. Mosk. Univ. Fiz. Astron.*, no. 3, p. 8, 1996.
3. M.P. Haugan and C.M. Will, *Physics Today,* vol. 40, p. 69, 1987.
4. C.M. Will, *Phys. Rev.,* vol. D45, p. 403, 1992.

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