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ON NONLINEAR WAVES DESCRIBED BY FIFTH-ORDER EVOLUTION EQUATIONS*

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The fifth-order quadratic-cubic nonlinear equation for long waves in the ocean is reduced to a nonlinear Schrödinger equation. Solutions in the form of breezer long-wave trains modulated by evolvent kinks are derived and studied. Long waves on finite-depth water are proved to be stable.

Intensive research is currently under way in the area of fifth-order nonlinear evolution equations. They describe, in particular, waves in cold quasineutral plasma [1], in electromagnetic transmission lines [2–4], gravity waves under an ice cover [5, 6], seismic waves in porous or fissured media [7], and long gravity waves on water in the second approximation [8] in the velocity potential expansion in parameters of amplitude ($\alpha = a/H$, where a is the amplitude and H is the depth) and frequency ($\beta = \lambda^2/H^2$, where λ is the wavelength) dispersions:

$$\eta_t + \eta_x + \frac{\beta}{6}\eta_{3x} + \frac{7\beta^2}{144}\eta_{5x} + \frac{3}{2}\alpha\eta\eta_x + \frac{5\alpha\beta}{12}\left[\eta\eta_{3x} + \frac{7}{2}\eta_x\eta_{2x}\right] - \frac{3a^2}{8}\eta^2\eta_x = 0. \tag{1}$$

To date, only two exact solutions to the fifth-order equations are known. They correspond to periodic stationary waves and a solitary fixed-amplitude wave [9, 10]. The fifth-order equations were studied numerically and analytically in [11–14] where, in particular, some self-similar solutions were found. In the present paper, we show that fifth-order equation (1) can be reduced to a cubically nonlinear Schrödinger equation to find its wave solutions and to explore their properties.

We make the substitution $x \to x - t$ and rewrite (1) as

$$\widehat{L} + \frac{3}{2}\alpha\eta(\partial_x\eta) + \frac{5\alpha\beta}{12}\eta(\partial_x^3\eta) + \frac{35}{24}\alpha\beta(\partial_x\eta)(\partial_x^2\eta) - \frac{3\alpha^2}{8}\eta^2(\partial_x\eta) = 0. \tag{2}$$

Here $\widehat{L} = L(\partial_t, \partial_x)$ is the linear differential operator, ∂_t and ∂_x denote partial derivatives. Putting $\partial_x = p$ and $\partial_t = q$, we have

$$L(p,q) = p + \frac{\beta}{6}q^3 + \frac{7\beta^2}{144}q^5. \tag{3}$$

Let us seek a solution to (2) in the form of an asymptotic series which contains a small parameter α,

$$\eta = \varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \dots . \tag{4}$$

Here $\varphi_0, \varphi_1, \dots$ are approximations of respective orders depending on the slow and fast variables x and t,

$$x \to x_0, X_1, X_2, \dots, X_N = \alpha^n x, \quad t \to t_0, T_1, T_2, \dots, T_N = \alpha^n t. \tag{5}$$

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When passing to the scale expansions of variables (5), the corresponding differential operators are transformed as

$$\partial_x \to \partial_{x_0} + \sum \alpha^k \partial_{X_k}, \quad \partial_t \to \partial_{t_0} + \sum \alpha^k \partial_{T_k}. \tag{6}$$

Substituting (4)–(6) into (2) and considering terms of the same orders, we obtain for $\varphi_0, \varphi_1, \ldots$ the following equations:

$$L_0\varphi_0 = 0, \tag{7}$$

$$L_0\varphi_1 + L_1\varphi_0 + \frac{3}{2}\varphi_0\partial_{x_0}\varphi_0 + \frac{5\beta}{12}\left[\varphi_0\partial^3_{x_0}\varphi_0 + \frac{7}{2}(\partial_{x_0}\varphi_0)(\partial^2_{x_0}\varphi_0)\right] = 0, \tag{8}$$

$$\begin{aligned} L_0\varphi_2 + L_1\varphi_1 + L_2\varphi_0 + \frac{3}{2}\left[\partial_{x_0}(\varphi_0\varphi_1) + \partial_{X_1}\left(\frac{\varphi_0^2}{2}\right)\right] + \frac{5\beta}{12}\left(\varphi_1\partial^3_{x_0}\varphi_0 + 3\varphi_0\partial^2_{x_0}\partial_{X_1}\varphi_0 + \varphi_0\partial^3_{x_0}\varphi_1\right) \\ + \frac{35\beta}{24}\left\{[(\partial_{X_1}\varphi_0) + (\partial_{x_0}\varphi_1)]\left(\partial^2_{x_0}\varphi_0\right) + (2\partial_{x_0}\partial_{X_1}\varphi_0 + \partial^2_{x_0}\varphi_1)(\partial_{x_0}\varphi_0) - \frac{3}{8}\varphi_0^2\partial_{x_0}\varphi_0\right\} = 0, \end{aligned} \tag{9}$$

where

$$L_0 = L(\partial_{t_0}, \partial_{x_0}),$$

$$L_1 = L_{0p}\partial_{T_1} + L_{0q}\partial_{X_1},$$

$$L_2 = L_{0p}\partial_{T_2} + L_{0q}\partial_{X_2} + L_{0pp}\partial^2_{T_1}/2 + L_{0pq}\partial_{T_1}\partial_{X_1} + L_{0qq}\partial^2_{X_1}/2$$

are the corresponding terms of the operator's Taylor expansions in the α-vicinity of $(\partial_{t_0}, \partial_{x_0})$,

$$L_{0p}n_q m \equiv \left.\frac{\partial^{n+m}}{\partial p^n \partial q^m}L(p,q)\right|_{p=\partial_{t_0},\, q=\partial_{x_0}}.$$

The solution's zero approximation φ_0 is sought for in the form of a harmonic wave with a slowly varying amplitude

$$\varphi_0 = A(X_1, \ldots, T_1, \ldots)\exp(i\theta) + \text{c.c.}, \quad \theta = kx - \omega t + \delta. \tag{10}$$

Substituting (10) into (7), we obtain the dispersion relation

$$L(-i\omega, ik) = 0 \leftrightarrow \omega = \omega(k). \tag{11}$$

Now, substituting (10)–(11) into (8), we get an equation for the function φ_1,

$$L_0\varphi_1 + (\partial_{T_1} + c_g\partial_{X_1})A\exp(i\theta) + \frac{3}{2}ikA^2\exp(2i\theta) + \frac{45}{24}(ik)^3\beta A^3\exp(2i\theta) + \text{c.c.} = 0, \tag{12}$$

where $c_g = d\omega/dk = c_g(k)$ is the group velocity. In order to avoid secularly growing terms in φ_1, we must demand that

$$(\partial_{T_1} + c_g\partial_{X_1})A = 0.$$

This implies that the water level first-order perturbations propagate at the group velocity

$$A = A(\theta_1, X_2, \ldots, T_2, \ldots), \quad \theta_1 = X_1 - c_gT_1. \tag{13}$$

The general solution to (12) can be represented in the form

$$\varphi_1 = A_{10} + A_{11}\exp(i\theta) + A_{12}\exp(2i\theta) + \text{c.c.} \tag{14}$$

For A_{12}, in the case of no resonance on the second harmonic, i.e., at $L(-2i\omega, 2ik) \neq 0$, we obtain from (12) and (14)

$$A_{12} = -\left[\frac{(3/2)ik + (45/24)\beta(ik)^3}{L(-2i\omega, 2ik)}\right]A^2 \equiv f(k)A^2. \tag{15}$$

Henceforth, without loss of generality, we may set $A_{11} = 0$, and determine A_{10} from (9) in the next order of smallness. By the same operational considerations (14), we seek solution for φ_2 in the form

$$\varphi_2 = A_{22}\exp(2i\theta) + A_{23}\exp(3i\theta) + \text{c.c.} \tag{16}$$

Substituting (10), (14), and (15) into (9) and satisfying the requirement that no secularly growing terms appear, we have to equate to zero all terms proportional to $\exp(i\theta)$ and 1. This leads to the following equations:

$$L_2 A + \left[\frac{3}{2}(ik) + \frac{5\beta}{12}(ik)^3\right] A A_{10} + \left[f(k)\frac{3}{2}(ik) - \frac{3}{8}(ik)\right] A^2 A^* = 0, \tag{17}$$

$$-c_g \partial_{\theta_1} A_{10} + \left[\frac{3}{2} + \frac{5}{4}(ik)^2 - \frac{35}{24}(ik)^3\beta\right] \partial_{\theta_1} A^2 = 0. \tag{18}$$

Integrating (18), we obtain a formula for A_{10},

$$A_{10} = \frac{1}{c_g}\left(\frac{3}{2} + \frac{5}{24}\beta k^2\right) A^2 + B(\tau). \tag{19}$$

Finally, substituting (19) into (17) yields the cubically nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \xi^2} + [\nu_1(\tau) + \nu(k)A^2]A = 0. \tag{20}$$

Here

$$\nu(k) = \frac{9k}{4\chi}\left[1 + \frac{47}{72}\chi + O(\chi^2)\right] > 0, \xi = \theta_1\left(\frac{-c_g}{2}\right)^{-1/2}, \quad \tau = \frac{1}{a}\left(T_2 - \frac{aX_2}{c_g}\right), \quad \chi = \beta k^2$$

with $\nu_1(\tau)$ and $a =$ const determined from the particular initial and boundary conditions.

Let us set for definiteness $\nu_1(\tau) =$ const, and $\mathrm{Im}\,\nu_1 = 0$. Then, substituting the expression

$$A = v(r)\exp[i(p\tau + q\xi)] \tag{21}$$

into (20), we can produce the following equation:

$$-d_{rr}v + (q^2 - p + \nu_1)v - i(c + 2q)d_r v + \nu v^3 = 0. \tag{22}$$

Here $v(r)$ is a real function of the variable $r = \xi - c\tau$, and c, p, and q are constant. Since v is a real function, we have $c + 2q = 0$. Integrating (22), we find

$$d_{rr}v = E - V(v), \tag{23}$$

where $V(v) = bv^2 - (\nu v^4/2)$ and $b = p - q^2 - \nu_1$. Since $\nu(k) > 0$, it follows that the potential $V(v)$ has a well and, consequently, (23) has a finite solution only if $b > 0$. This solution can be expressed in terms of the elliptic Jacobi function $\mathrm{sn}(n/m)$. Introducing a more convenient notation, the final result can be represented by

$$A = A_0\,\mathrm{sn}\left[\left(\frac{A_0}{m}\right) - \left(\frac{\nu}{2m}\right)^{1/2}(r - r_0)\right]\exp\left[i\left(p\tau - \frac{c\xi}{2}\right)\right] = R\exp(i\Phi). \tag{24}$$

Here A_0, m, c and r_0 are intermediate real parameters, and

$$p = \frac{\nu}{2}A_0^2\left[1 + \left(\frac{1}{m}\right)^2\right] + \frac{c^2}{4} + \nu_1.$$

Algebraic equation (24) describes a breezer wave train. The train consists of waves with the first harmonic's amplitude R modulated by an elliptic sine of order m, which is determined by the initial conditions. The evolvent amplitude R is periodically oscillating with respect to the variable

$$\tau = \frac{\alpha}{\sqrt{-c_g^1(k)/2}}(x - c_g t) - \alpha^2 c\left[\frac{t - (a/c_g)x}{a}\right],$$

which in the first order is associated with a system of waves propagating at the group velocity $c_g(k)$. The spatial period of oscillations in this system is

$$\lambda = 4K(m)\left[-\frac{c_g'(k)}{2}\right]^{1/2} \Big/ \left[A_0\alpha\left(\frac{\nu}{2m}\right)^{1/2}\right], \tag{25}$$

where $K(m)$ is the complete elliptic integral of the first kind.

As $m \to 0$, the breezer train amplitude is modulated by a harmonic wave

$$\lim_{m\to 0} R \to A_0 \sin\left[\frac{2\pi}{\lambda}(x - c_g t) + O(\alpha^2)\right] + O(m^2). \tag{26}$$

For a certain choice of $m = m(\alpha, \beta, k)$, this period may be finite, i.e., have a scale of the order of unity instead of $O(1/\alpha)$. As $m \to 1$, the breezer train amplitude is modulated by a switch wave, or a kink, with an infinitely large period,

$$\lim_{m\to 1} R \to A_0 \tanh\left\{A_0\left[\frac{\nu(k)}{-c_g'(k)}\right]^{1/2}\alpha(x - c_g t) + O(\alpha^2)\right\}. \tag{27}$$

The derivative of kink (27) is a soliton. In the general case, the shape of the long-wave breezer train on finite-depth water depends on the wavelength λ (25) and the amplitude A_0. This is a transitional form between the harmonic (26) and the switch wave, or kink, (27).

It is essential that, along with breezer-type solutions (21), (24), equation (20) also allows for conventional wave solutions

$$A = A_0 \exp(i\varphi_0), \quad \varphi_0 = k\xi - \omega t, \tag{28}$$

which turn out to be stable with respect to amplitude and phase perturbations. In other words, in our case of long waves on finite-depth water there is no modulation instability, typical for gravity waves on deep water (Zakharov–Benjamen–Fair instability [15]). This nontrivial statement, which actually asserts that long waves on water are stable in the second approximation, will be proved by considering small additions to solution (28),

$$A = A_0 + \delta, \quad \varphi = \varphi_0 + \varphi_1. \tag{29}$$

Substituting (29) into (20) and retaining only linear terms in δ and φ_1, we obtain the following system of equations with respect to perturbations:

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau} - 2k\frac{\partial}{\partial\xi}\right)\delta - A_0\frac{\partial^2\varphi_1}{\partial\xi^2} = 0, \\ &\left(\omega - \frac{\partial^2}{\partial\xi^2} + k^2 + \nu_1 + 3\nu A_0^2\right)\delta + A_0\left(2k\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\tau}\right)\varphi_1 = 0. \end{aligned} \tag{30}$$

Seeking a solution to (30) in the form of waves δ, $\varphi_1 \sim \exp[i(\varkappa\xi - \Omega\tau)]$, we obtain the dispersion relation

$$\Omega = 2\varkappa k \pm \sqrt{\varkappa^2 + 2\nu A_0^2} \tag{31}$$

which, because ν is positive, implies that Ω is real, proving our statement. Formula (31) describes a Bogolyubov spectrum of excitations in a medium that is moving at the velocity ω/k, but we are not going to discuss the corresponding physical analogy here.

The results produced in this study are important for the theory of long waves on water, because the properties of such waves are established in the second approximation. We have shown that these waves are stable and can be studied with the aid of a nonlinear Shrödinger equation. The breezer solutions discovered can explain, for instance, tsunami propagation in the ocean in the form of wave trains with a time-varying evolvent. Due to this fact, in particular, tsunami waves may approach the shore at both negative and positive level variation phases. One can find many examples of such water level variation records in existing tsunami catalogs.

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