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ASTRONOMY

ECCENTRICITY VARIATIONS OF THE EXTERNAL ORBIT OF ξ U.Ma.

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An analytical solution for variations of the external orbit in ternary stellar systems of ε Lyr type is obtained. An example of the ξ U.Ma. system is considered.

INTRODUCTION

In [1] a system of differential equations was derived to describe perturbed motion of a ternary system, where the Hamiltonian third- and fourth-order terms are taken into account. We used a method of variation of arbitrary constants and introduced a new system of canonical variables. Higher-order terms may play an important role in the case of origination of resonances. Harrington [2] treated the effect of higher-order terms by numerical methods. However, this proved to take considerable machine time, and it was therefore desirable to develop an analytical method for taking perturbations into account.

In the present paper, allowing for the third term in the Hamiltonian, we obtained analytical expressions of the external orbit eccentricity perturbations. In the intermediate motion, the remote point orbit is a non-Keplerian ellipse with moving node, periastron, and invariable eccentricity. For perturbed motion, taking into account the third term, the Hamiltonian depends on two angular variables g_1 and g_2, periastrons of the external and internal orbits. In this case the eccentricity of the external orbit has weak periodic variations. The ξ U.Ma. system will be considered as an example.

1. STATEMENT OF THE PROBLEM

Let us consider the motion of three material points with comparable masses m_0, m_1, m_2 and with the semimajor axes ratio $a_1/a_2 < 0.1$. We shall call the orbit of the close pair internal and the orbit of the remote point external. The osculating orbits of the ternary system components are assumed elliptical with arbitrary eccentricities and inclinations. A perturbing function from which the short-period terms are eliminated has the form

$$
\begin{aligned}
R = {} & \frac{15}{512}\gamma_4 \frac{L_1^6}{L_2^8} \frac{e_1 e_2 \sqrt{1-e_2^2}}{(1-e_2)^3 (1+e_2)^3} \\
& \times \Big[\left(4+3e_1^2\right)\left(-1-11q+5q^2+15q^3\right)\cos\left(g_1-g_2\right) + 35e_1^2(1-q)(1+q)^2\cos\left(3g_1-g_2\right) \\
& + \left(4+3e_1^2\right)\left(-1+11q+5q^2-15q^3\right)\cos\left(g_1+g_2\right) + 35e_1^2(-1+q)^2(1+q)\cos\left(3g_1+g_2\right)\Big],
\end{aligned} \tag{1}
$$

where γ_4 is the mass-dependent coefficient; L_1 and L_2 are the Delaunay elements; e_1 and e_2 are the eccentricities of internal and external orbits; q is the cosine of mutual inclination angle; g_1 and g_2 are the variables of the orbit periastrons. A canonical system of equations in variables Λ_j and λ_j has the form

$$
\frac{d\Lambda_j}{dt} = \frac{\partial Z}{\partial \lambda_j}, \quad \frac{d\lambda_j}{dt} = -\frac{\partial Z}{\partial \Lambda_j}, \tag{2}
$$

for $j = 1, 2, 3, 4, 5$, where

$$Z = \varepsilon + R; \tag{3}$$

$$\Lambda_1 = A_1, \quad \Lambda_2 = A_2, \quad \Lambda_3 = \frac{2}{\pi} W_1(g_1),$$
$$\Lambda_4 = A_4, \quad \Lambda_5 = A_5; \tag{4}$$

$$\lambda_3 = \lambda_{30} + \nu_3 (t - t_0); \tag{5}$$

$$\lambda_{30} = \frac{3\pi\delta}{A_1 \overline{G}_2^2 K \Sigma_1} B_3; \quad \nu_3 = -\frac{3\pi\delta}{A_1 \overline{G}_2^2 K \Sigma_1} \frac{\partial \varepsilon}{\partial A_3};$$

$$\lambda_4 = B_4 + \kappa_4 (t - t_0) + (Q_4 \Sigma_1 + Q_5 \Sigma_2 + Q_6 \Sigma_4 + Q_7 \Sigma_5) \frac{2K}{\pi} \lambda_3; \tag{6}$$

$$W_1 = \int_{-\pi/2}^{0} W_1'(g_1)\, dg_1; \tag{7}$$

$W_1(g_1)$ is the function which in the intermediate motion [3] satisfies a sixth-order equation; λ_3 and λ_4 are the secular parts of the angular variables g_1 and g_2; A_j, B_3, and B_4 are the integration constants in the intermediate motion.

The Hamiltonian of the system Z has two terms ε and R. The term ε corresponds to unperturbed motion and does not depend on angular variables, R is the perturbing part dependent on two angular variables, g_1 and g_2. It is necessary to express the perturbing function in terms of new variables Λ_j, λ_3, and λ_4. Expressing the function R in terms of Λ_3 is a complicated problem since it requires that the equation

$$\Lambda_3 = \frac{2}{\pi} W_1(g_1) \tag{8}$$

should be solved.

However, it is possible to proceed as follows. We find the total derivative

$$\frac{d\Lambda_3}{dt} = \frac{2}{\pi}\left(\frac{\partial W_1}{\partial A_1}\frac{dA_1}{dt} + \frac{\partial W_1}{\partial A_2}\frac{dA_2}{dt} + \frac{\partial W_1}{\partial A_3}\frac{dA_3}{dt} + \frac{\partial W_1}{\partial A_4}\frac{dA_4}{dt} + \frac{\partial W_1}{\partial A_5}\frac{dA_5}{dt}\right), \tag{9}$$

where

$$\frac{dA_1}{dt} = \frac{\partial Z}{\partial \lambda_1} = 0, \quad \frac{dA_2}{dt} = \frac{\partial Z}{\partial \lambda_2} = 0, \quad \frac{dA_5}{dt} = \frac{\partial Z}{\partial \lambda_5} = 0. \tag{10}$$

Whence A_1, A_2, and A_5 are constant.

From the formulas of intermediate motion [3] we have

$$\frac{\partial W_1}{\partial A_3} = \frac{A_1 \overline{G}_2^2}{6\delta} K \Sigma_1,$$
$$\frac{\partial W_1}{\partial A_4} = -K (Q_4 \Sigma_1 + Q_5 \Sigma_2 + Q_6 \Sigma_4 + Q_7 \Sigma_5), \tag{11}$$

where K is the complete elliptical integral. It is easy to obtain equations

$$\frac{dA_3}{dt} = \frac{6\delta}{A_1 \overline{G}_2^2 \Sigma_1}\left[\frac{\pi}{2K}\frac{\partial R}{\partial \lambda_3} + (Q_4 \Sigma_4 + Q_5 \Sigma_2 + Q_6 \Sigma_4 + Q_7 \Sigma_5)\frac{\partial R}{\partial \lambda_4}\right],$$
$$\frac{dA_4}{dt} = \frac{\partial R}{\partial \lambda_4}. \tag{12}$$

Let us transform equations

$$\frac{d\lambda_i}{dt} = -\frac{\partial Z}{\partial \Lambda_i} \tag{13}$$

as follows:

$$\frac{\partial Z}{\partial \Lambda_i} = \frac{\partial Z}{\partial A_1}\frac{\partial A_1}{\partial \Lambda_i} + \frac{\partial Z}{\partial A_2}\frac{\partial A_2}{\partial \Lambda_i} + \frac{\partial Z}{\partial A_3}\frac{\partial A_3}{\partial \Lambda_i} + \frac{\partial Z}{\partial A_4}\frac{\partial A_4}{\partial \Lambda_i} + \frac{\partial Z}{\partial A_5}\frac{\partial A_5}{\partial \Lambda_i}, \tag{14}$$

where

$$\begin{array}{lllll}
\frac{\partial A_1}{\partial \Lambda_1} = 1, & \frac{\partial A_2}{\partial \Lambda_1} = 0, & \frac{\partial A_3}{\partial \Lambda_1} = \frac{\partial W_1}{\partial A_1} \Big/ \frac{\partial W_1}{\partial A_3}, & \frac{\partial A_4}{\partial \Lambda_1} = 0, & \frac{\partial A_5}{\partial \Lambda_1} = 0, \\
\frac{\partial A_1}{\partial \Lambda_2} = 0, & \frac{\partial A_2}{\partial \Lambda_2} = 1, & \frac{\partial A_3}{\partial \Lambda_2} = \frac{\partial W_1}{\partial A_2} \Big/ \frac{\partial W_1}{\partial A_3}, & \frac{\partial A_4}{\partial \Lambda_2} = 0, & \frac{\partial A_5}{\partial \Lambda_2} = 0, \\
\frac{\partial A_1}{\partial \Lambda_3} = 0, & \frac{\partial A_2}{\partial \Lambda_3} = 0, & \frac{\partial A_3}{\partial \Lambda_3} = \frac{\pi}{2} \Big/ \frac{\partial W_1}{\partial A_3}, & \frac{\partial A_4}{\partial \Lambda_3} = 0, & \frac{\partial A_5}{\partial \Lambda_3} = 0, \\
\frac{\partial A_1}{\partial \Lambda_4} = 0, & \frac{\partial A_2}{\partial \Lambda_4} = 0, & \frac{\partial A_3}{\partial \Lambda_4} = \frac{\partial W_1}{\partial A_4} \Big/ \frac{\partial W_1}{\partial A_3}, & \frac{\partial A_4}{\partial \Lambda_4} = 1, & \frac{\partial A_5}{\partial \Lambda_4} = 0, \\
\frac{\partial A_1}{\partial \Lambda_5} = 0, & \frac{\partial A_2}{\partial \Lambda_5} = 0, & \frac{\partial A_3}{\partial \Lambda_5} = \frac{\partial W_1}{\partial A_5} \Big/ \frac{\partial W_1}{\partial A_3}, & \frac{\partial A_4}{\partial \Lambda_5} = 0, & \frac{\partial A_5}{\partial \Lambda_5} = 1.
\end{array} \tag{15}$$

Partial derivatives have the form [3]

$$\begin{aligned}
\frac{\partial W_1}{\partial A_1} &= -K\,\left(Q_1\,\Sigma_1 + Q_2\,\Sigma_2 + Q_3\,\Sigma_3\right), \\
\frac{\partial W_1}{\partial A_2} &= 0, \\
\frac{\partial W_1}{\partial A_5} &= -K\,\left(\frac{4\,\bar{c}\,\overline{G}_2^2}{\delta}\,\Sigma_1 + Q_6\,\Sigma_4 - Q_7\,\Sigma_5\right).
\end{aligned} \tag{16}$$

We obtain

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial Z}{\partial A_1} - \frac{\partial Z}{\partial A_3}\left[-\frac{Q_1\,\Sigma_1 + Q_2\,\Sigma_2 + Q_3\,\Sigma_3}{A_1\,\overline{G}_2^2\,\Sigma_1}\,6\,\delta\right], \\
\frac{d\lambda_2}{dt} &= -\frac{\partial Z}{\partial A_2}, \\
\frac{d\lambda_3}{dt} &= -\left(\frac{6\,\pi\,\delta}{2\,A_1\,\overline{G}_2^2\,K\,\Sigma_1}\right)\frac{\partial Z}{\partial A_3}, \\
\frac{d\lambda_4}{dt} &= \left[\frac{Q_4\,\Sigma_1 + Q_5\,\Sigma_2 + Q_6\,\Sigma_4 + Q_7\,\Sigma_5}{A_1\,\overline{G}_2^2\,\Sigma_1}\,6\,\delta\right]\frac{\partial Z}{\partial A_3} - \frac{\partial Z}{\partial A_4}, \\
\frac{d\lambda_5}{dt} &= \left[\frac{\frac{4\,\bar{c}\,\overline{G}_2^2}{\delta}\,\Sigma_1 + Q_6\,\Sigma_4 - Q_7\,\Sigma_5}{A_1\,\overline{G}_2^2\,\Sigma_1}\,6\,\delta\right]\frac{\partial Z}{\partial A_3}.
\end{aligned} \tag{17}$$

Considering that the constants A_j can easily be expressed in terms of the osculating Keplerian elements, we obtain a differential equation for the external orbit eccentricity variation. The differential equation for A_4 is

$$\frac{dA_4}{dt} = \frac{\partial R}{\partial \lambda_4}. \tag{18}$$

The constant A_4 is expressed in terms of the osculating Keplerian elements as follows:

$$A_4 = L_2 \sqrt{1 - e_2^2}. \tag{19}$$

Then

$$\frac{dA_4}{dt} = -L_2 \frac{e_2}{\sqrt{1-e_2^2}} \frac{de_2}{dt}. \tag{20}$$

The differential equation for e_2 has the form

$$\frac{de_2}{dt} = -\frac{\sqrt{1-e_2^2}}{L_2\, e_2} \frac{\partial R}{\partial \lambda_4} = -\frac{A_4}{A_2^2\, e_2} \frac{\partial R}{\partial \lambda_4}, \tag{21}$$

where R is the perturbing function in new variables:

$$R = \frac{15\, A_1^6\, e_1\, e_2\, \sqrt{1-e_2^2}\, \gamma_4}{512\, A_2^8\, (1-e_2)^3\, (1+e_2)^3}$$
$$\times \big[\left(4 + 3e_1^2\right) \left(-1 - 11q + 5q^2 + 15q^3\right) \cos\left(\lambda_3 - \lambda_4 + \Theta\right) + 35\, e_1^2\, (1-q)\, (1+q)^2 \cos\left(3\, \lambda_3 - \lambda_4 + \Theta\right)$$
$$+ \left(4 + 3\, e_1^2\right) \left(-1 + 11\, q + 5\, q^2 - 15\, q^3\right) \cos\left(\lambda_3 + \lambda_4 + \Theta\right) + 35\, e_1^2\, (-1+q)^2\, (1+q) \cos\left(3\, \lambda_3 + \lambda_4 + \Theta\right) \big]. \tag{22}$$

Solving equation (21) by the method of successive approximations, we obtained a first approximation for the eccentricity variation:

$$\delta_1 e_2 = \frac{15 A_1^6 e_1 \left(1 - e_2^2\right) \gamma_4}{512 A_2^9 (1-e_2)^3 (1+e_2)^3}$$
$$\times \left[-\frac{35 e_1^2 (1-q)(1+q)^2}{3\nu_3 - \nu_4} \cos\left(t\left(-3\nu_3 + \nu_4\right) + \Theta\right) \right.$$
$$- \frac{\left(4 + 3e_1^2\right)\left(-1 - 11q + 5q^2 + 15q^3\right)}{\nu_3 - \nu_4} \cos\left(t\left(-\nu_3 + \nu_4\right) + \Theta\right)$$
$$+ \frac{35 e_1^2 (-1+q)^2 (1+q)}{3\nu_3 + \nu_4} \cos\left(t\left(3\nu_3 + \nu_4\right) + \Theta\right)$$
$$\left. + \frac{\left(4 + 3e_1^2\right)\left(-1 + 11q + 5q^2 - 15q^3\right)}{\nu_3 + \nu_4} \cos\left(t\left(\nu_3 + \nu_4\right) + \Theta\right) \right]. \tag{23}$$

Here Θ depends on the initial conditions.

2. APPLICATION OF THE THEORY TO THE ξ U.Ma. SYSTEM

The ternary stellar system ξ U.Ma. whose components move in short-period orbits with periods of about two years (internal orbit) and 60 years (external orbit) is an interesting object for applying this theory. The system has been followed up for several decades. It has masses of the same order of magnitude as the Sun mass, and all the Keplerian elements required for the calculations are known for it. We took the following elements [4] for the epoch $T_0 = 1900.00$:

$$m_0 = 0.83, \quad m_1 = 0.30, \quad m_2 = 0.92,$$

Internal orbit:	External orbit:
$a_1 = 1.56$ AU	$a_2 = 19.46$ AU
$e_1 = 0.56$	$e_2 = 0.414$
$M_1 = 118.29°$	$M_2 = 211.58°$
$\omega_1 = 146.00°$	$\omega_2 = 127.5°$
$\Omega_1 = 326.00°$	$\Omega_2 = 101.5°$
$i_1 = 86.3°$	$i_2 = 122.65°$

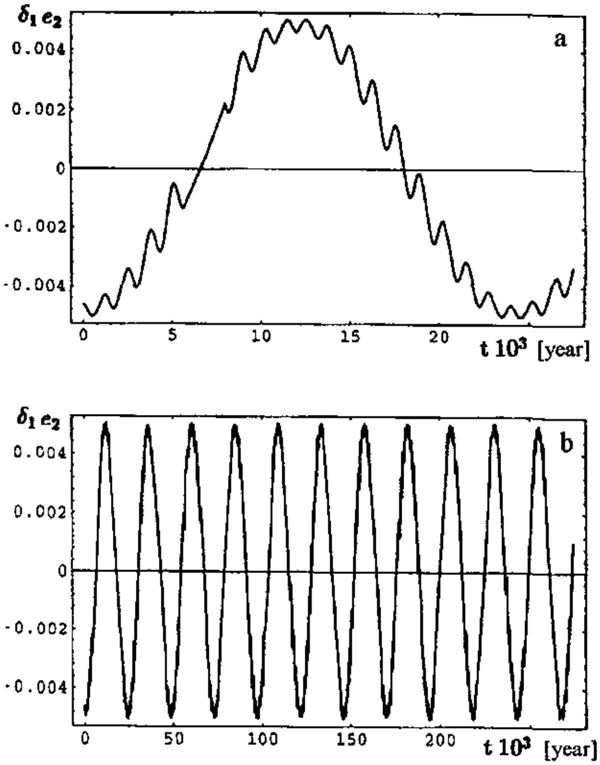

Fig. 1

Eccentricity variations $\delta_1 e_2$ in the interval (*a*) 27 000 years and (*b*) 270 000 years.

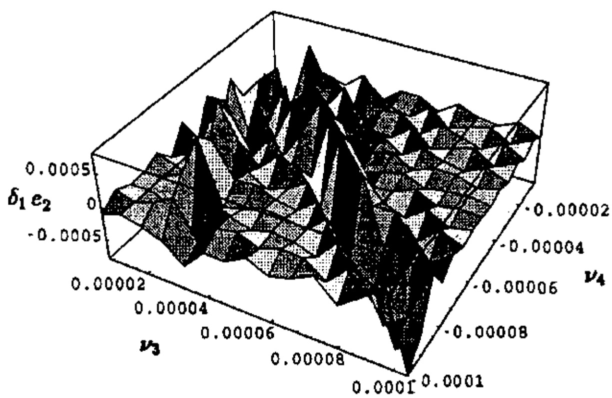

Fig. 2

Eccentricity variations $\delta_1 e_2$ as ν_3 varies from 0.645×10^{-5} to 1×10^{-4} and ν_4 from -0.716×10^{-5} to 1×10^{-4} in the interval $t = 270\,000$ years.

The results are presented graphically. Figure 1 illustrates weak variations within the time intervals of 27 000 years and 270 000 years. Figure 2 illustrates phenomena close to resonances.

CONCLUSION

In this paper we derived the differential equations for determination of eccentricity perturbations, when in the Hamiltonian the terms of the third order of smallness are taken into account. Example calculations are performed for the ternary stellar system ξ U.Ma. It is shown that in this case the eccentricity of the external orbit may have weak long-period variations and resonance-like phenomena may arise.

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