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BRIEF COMMUNICATIONS
THEORETICAL AND MATHEMATICAL PHYSICS
CALCULATION OF EIGENVALUES OF THIRD BOUNDARY-VALUE PROBLEMS FOR THE BESSEL EQUATION

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The eigenvalues of the Sturm–Liouville problems for the Bessel equation with boundary conditions of the third kind have been calculated by the differential-parametric method.

1. Solving the eigenvalue problem for the disk of radius $0 < r \leq a$ with the boundary condition (generally not self-conjugate) of the third kind leads to the suggestion that the Sturm–Liouville problem for the Bessel equation should be considered:

$$L[R] + \lambda^2 r R(r) = 0 \quad (0 < r < a) \tag{1}$$

with the condition that the solution is bounded in the disk center,

$$|R(0)| < \infty \tag{2}$$

and

$$R'(a) + hR(a) = 0 \tag{3}$$

on the disk boundary, where

$$L[R] = (rR')' - (n^2/r^2)R,$$

h is the specified number (possibly a complex one, $\operatorname{Re} h > 0$), λ are the eigenvalues sought, and $R(r)$ are the eigenfunctions.

Consider the calculation algorithm for the eigenvalues that is based on reducing the problem to the Cauchy problem and does not require any calculation of the eigenfunctions.

Introducing a new variable by putting

$$x = \lambda r$$

and denoting

$$R(r) = R(x/\lambda) = y(x),$$

we write down the Bessel equation in the form [1, 2],

$$(xy')'/x + (1 - n^2/x^2)y(x) = 0 \quad (0 < x < \lambda a). \tag{1'}$$

The boundary conditions take the following form:

$$|y(0)| < \infty, \tag{2'}$$

$$\lambda y'(\lambda a) + hy(\lambda a) = 0. \tag{3'}$$

The resulting problem defines implicitly the eigenvalues λ as functions of the parameter h. We find them by the differential-parametric method [3].

Introduce the function

$$S(z) = y'(z)/y(z)$$

and rewrite equation (3′) as

$$S(\lambda a) = -h/\lambda. \tag{4}$$

Since $y(z)$ satisfies Bessel equation (1′), the function $S(z)$ has the differential-polynomial property (DP-property), i.e., its derivative is expressed in the form of the polynomial of the function proper:

$$S'(z) = (n^2/z^2 - 1) - S(z)/z - S^2(z). \tag{5}$$

Applying the theorem of the derivative of an implicit function to equation (4) and making use of the DP-property (5), we arrive at the Cauchy problem:

$$d\lambda/dh = \lambda a/\left[(\lambda a)^2 + (ha)^2 - n^2\right], \tag{6}$$

or, denoting $\lambda_a = \lambda a$ and $\alpha = ha$, we come to the problem

$$d\lambda_a/d\alpha = \lambda_a/\left(\lambda_a^2 + \alpha^2 - n^2\right), \quad \lambda_a\Big|_{\alpha=0} = \nu a,$$

where ν are the eigenvalues of the corresponding self-conjugate (at $h = 0$) problem (1)–(3) with the boundary condition of the second kind, i.e., ν are zeros of the Bessel function derivative, which are assumed to be known [4].

Thus, under the condition

$$(\lambda a)^2 + (ha)^2 \neq n^2,$$

the eigenvalue spectrum for the boundary-value problem (1)–(3) is nondegenerate and is found by solving the Cauchy problem (6)–(7).

Under the condition $|h| \ll 1$, $|\lambda a| \gg n$, the asymptotic behavior of eigenvalues is written as:

$$\lambda \simeq \nu + h/(\nu a).$$

2. Solving the eigenvalue problem for the ring $0 < b \le r \le a$ with the boundary conditions of the third kind leads to the analogous Sturm–Liouville problem for the Bessel equation:

$$L[R] + \lambda^2 r R(r) = 0 \quad (0 < b < r < a)$$

with the condition at the ring inside boundary

$$R'(b) - h_1 R(b) = 0 \quad (\mathrm{Re}\, h_1 > 0)$$

and the condition at the ring outside boundary

$$R'(a) + h_2 R(a) = 0 \quad (\mathrm{Re}\, h_2 > 0).$$

Let the solution of the corresponding (at $h_1 = h_2 = 0$) self-conjugate eigenvalue problem with the boundary conditions of the second kind be known and equal to $\varkappa$.

Then, by analogy with what has been previously made, we apply the DP-method to reduce the eigenvalue problem for the Bessel equation

$$L[R] + \mu^2 r R(r) = 0 \quad (0 < b < r < a)$$

with the second-kind condition at the ring inside boundary ($h_1 = 0$),

$$R'(b) = 0,$$

and the third-kind condition at the ring outside boundary,

$$R'(a) + h_2 R(a) = 0,$$

to the Cauchy problem ($\lambda_a = \mu a$, $\alpha = h_2 a$):

$$d\lambda_a/d\alpha = \lambda_a/\left(\lambda_a^2 + \alpha^2 - n^2\right), \quad \lambda_a\Big|_{\alpha=0} = \varkappa a.$$

Upon finding its solution, $\lambda_a = \mu a$, the eigenvalues λ of the initial problem are calculated with the help of the following Cauchy problem obtained ($\lambda_b = \lambda b$, $\beta = -h_1 b$):

$$d\lambda_b/d\beta = \lambda_b/\left(\lambda_b^2 + \beta^2 - n^2\right), \quad \lambda_b\Big|_{\beta=0} = \lambda_a = \mu a.$$

In conclusion, we notice that the method of calculating the eigenvalues by reduction to the Cauchy problems enables us to find the eigenvalues without using the eigenfunctions, whose calculation often presents considerable difficulties (especially for complex h, h_1, and h_2 and for noninteger or complex n).

The problems similar to those considered above have also to be solved when eigenvalues of the third boundary-value problems for a sphere or circular sector etc. are to be calculated [2]. An example is the investigation of the wave-resonance coaxial [5], spherical and other systems with the Leontovich-type impedance boundary conditions.

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