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THEORETICAL AND MATHEMATICAL PHYSICS

POSSIBILITY-THEORETIC METHOD OF MEASUREMENT REDUCTION

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A possibility-theoretic model has been applied to solve the problem of reducing a measurement to an ideal instrument.

INTRODUCTION

Consider an experimental measurement

$$\xi = Af + \nu, \tag{1}$$

which is the corrupted by noise $\nu \in \mathcal{R}$ output signal Af of an instrument A that has received a signal $f \in \mathcal{F}$ from the object being measured and the environment. Suppose that $Uf \in \mathcal{U}$ are the parameters of the object under examination, A: $\mathcal{F} \to \mathcal{R}$ and U: $\mathcal{F} \to \mathcal{U}$ are the operators modeling respectively, the measuring instrument and the dependence of the signal f supplied by the measured object and the environment on the parameters of the studied object not disturbed by measurement; here, $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{U}$ are the finite-dimensional Euclidean spaces [1]. This paper is concerned with interpreting measurement (1); the problem is to determine an estimation (interpretation) strategy $d(\cdot)$: $\mathcal{R} \to \mathcal{U}$ such that the element $d(\xi)$ can be considered the most precise version of the value Uf of the object parameters.

The operator U models what, in experimental studies, is usually called an ideal measuring instrument, it generates the parameters corresponding to the state of the object not distorted by measurement. The problem under consideration is called the problem of measurement reduction to an ideal instrument [1].

Solving the reduction problem requires the specification of a model of measurement (1). In this paper, we solve the reduction problem for possibility-theoretic models similar to the probability-theoretic models $[A, \Sigma]$ and $[A, F, f_0, \Sigma]$, where the noise ν is a random element of $\mathcal{R}$ with expectation $\mathbf{E}\nu = 0$ and covariance operator Σ: $\mathcal{R} \to \mathcal{R}$. In the model $[A, \Sigma]$, f is assumed to be an *a priori* arbitrary element of $\mathcal{F}$, while in the model $[A, F, f_0, \Sigma]$, f is a random element of $\mathcal{F}$, $\mathbf{E}f = f_0$, and F: $\mathcal{F} \to \mathcal{F}$ is the covariance operator of f [1].

The reduction problems for these models are solved in [1–3].

1. REDUCTION OF MEASUREMENT (1) FOR AN *A PRIORI* ARBITRARY SIGNAL f

Let ν be a fuzzy element of $\mathcal{R}$ with a distribution $\varphi^{\nu}(\cdot)$: $\mathcal{R} \to [0,1]$; then ξ in (1) is also a fuzzy element of $\mathcal{R}$, and $\varphi^{\xi}(x, f) = \varphi\nu(x - Af)$, where $x \in \mathcal{R}$, is the distribution of ξ dependent on $f \in \mathcal{F}$ as on a parameter. In this section, we assume that f is an *a priori* arbitrary element of $\mathcal{F}$ [4]. Let us introduce a fuzzy error relation $(\mathcal{U} \times \mathcal{U}, l(\cdot,\cdot))$, where $l(Uf, u)$ is the possibility of the error due to the choice of $u \in \mathcal{U}$ as the parameter set Uf for each value $f \in \mathcal{F}$ of the input signal [5].

The quality of a strategy $d(\cdot)\colon \mathcal{R} \to \mathcal{U}$ for estimating the function Uf of the parameter $f \in \mathcal{F}$ of the distribution $\varphi^{\xi}(x, f) = \varphi^{\nu}(x - Af)$, where $x \in \mathcal{R}$ is characterized by the necessity of estimation error [5]

$$M(d(\cdot)) = \neg \sup_{x\in\mathcal{R}} \sup_{f\in\mathcal{F}} \min(\varphi^{\nu}(x - Af), \neg l(Uf, d(x))). \tag{2}$$

An N-optimal strategy $d^*(\cdot)$ is determined by the condition

$$M(d^*(\cdot)) = \min_{d(\cdot)\colon\, \mathcal{R}\to\mathcal{U}} M(d(\cdot)).$$

The problem of finding an N-optimal strategy is similar to the problems of N-optimal estimation considered in [4], where it was shown that an N-optimal strategy $d^*(x)$, $x \in X$, can be found by solving a simpler problem

$$\neg M(d, x) = \sup_{f\in\mathcal{F}} \min(\varphi^{\nu}(x - Af), \neg l(Uf, d)) \sim \max_{d\in\mathcal{U}} \tag{3}$$

for every $x \in \mathcal{R}$.

Consider an important special case of problem (3), where

$$l(u, v) = l^0(u, v) = \begin{cases} > 0 & \text{if } u \neq v, \\ 0 & \text{if } u = v, \end{cases} \quad u, v \in \mathcal{U}. \tag{4}$$

Since $\max_{d\in\mathcal{U}} \neg l(Uf, d) = 1$ is attained for the unique element $d = Uf$, we have $\max_{d\in\mathcal{U}} \sup_{f\in\mathcal{F}} \min(\varphi^{\nu}(x - Af), \neg l(Uf, x)) = \sup_{f\in\mathcal{F}} \min(\varphi^{\nu}(x - Af), \max_{d\in\mathcal{U}} \neg l(Uf, d)) = \sup_{f\in\mathcal{F}} \varphi^{\nu}(x - Af)$. Therefore, $d^*(x) = Uf(x)$, where $f(x)$ is the estimate of f with maximum possibility [5]: $\varphi^{\nu}(x - Af(x)) = \max_{f\in\mathcal{F}} \varphi^{\nu}(x - Af)$.

Let $\varphi^{\nu}(\cdot) = \rho(\|\Sigma^{-1/2} \cdot\|)$, where $\Sigma\colon \mathcal{R} \to \mathcal{R}$ is a positive definite operator (an analog of the correlation operator of measurement error in the model $[A, \Sigma]$) and $\rho(\cdot)\colon [0, \infty) \to [0, 1]$ is a strictly monotone decreasing continuous function specifying the distribution of the fuzzy error vector $\nu \in \mathcal{R}$ and normalized by $\rho(0) = 1$. Under these conditions, problem (3) is equivalent to the problem of finding the maximal possibility estimate [5] of the parameter $f \in \mathcal{F}$ for $\xi = x$ by solving the minimization problem

$$\|\Sigma^{-1/2}(x - Af)\| \sim \min_{f\in\mathcal{F}}, \tag{5}$$

and setting $d^*(x) = Uf(x)$, where $f(x)$ is the value of f, at which the minimum in (5) is attained. Put $\Sigma^{-1/2}x = y$ and $\Sigma^{-1/2}A = B$; let B^- be the operator pseudoinverse of B [6]. For any $f \in \mathcal{F}$,

$$\|y - Bf\|^2 = \|B(B^- y - f)\|^2 + \|(I - BB^-)y\|^2 \geq \|(I - BB^-)y\|^2, \tag{6}$$

and the equality in (6) is attained at* $f = B^- y + b$, where b is an arbitrary vector from the kernel $\mathcal{N}(B)$ of the operator B (i.e., such that $Bb = 0$) [6]. Therefore, the minimum in (5) is attained at $f = f(x) = (\Sigma^{-1/2}A)^-\Sigma^{-1/2}x + b$, and

$$d^* = d^*(x) = Uf(x) = U(\Sigma^{-1/2}A)^-\Sigma^{-1/2}x + Ub. \tag{7}$$

If $\mathcal{N}(B) \subset \mathcal{N}(U)$, then $Ub = 0$ for any $b \in \mathcal{N}(B) = \mathcal{N}(A)$, and equality (7) specifies a unique N-optimal strategy; according to this strategy, the N-optimal estimate $d^*(\xi)$ of the element Uf is the fuzzy element

$$d^*(\xi) = U(\Sigma^{-1/2}A)^-\Sigma^{-1/2}\xi. \tag{8}$$

It is of interest that the same formula gives the best mean-square linear minimax estimate of Uf if the noise ν in the equality $\xi = Af + \nu$ is a random element of $\mathcal{R}$ with zero expectation and correlation operator Σ [1].

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* B^- and B^* are the operators pseudoinverse of B and adjoint of B, respectively.

If the equality $Bb = 0$ does not imply that $Ub = 0$, then the optimal strategy $d^*(\cdot)$ is not unique

$$d^*(\xi) = U(\Sigma^{-1/2}A)^-\Sigma^{-1/2}\xi + Ua(\xi), \tag{9}$$

where $a(\cdot)$: $\mathcal{R} \to \mathcal{N}(A)$ is an arbitrary function with values in $\mathcal{N}(A)$ such that $Aa(x) = 0$ for $x \in \mathcal{R}$. Under these conditions, there exists no stochastic analog of estimate (9) [6].

For strategies (8) and (9), the necessity of error is

$$M(d^*(\cdot)) = \neg \sup_{x\in X} \rho(\|(I - \Sigma^{-1/2}A(\Sigma^{-1/2}A)^-)\Sigma^{-1/2}x\|) = 0.$$

The supremum here is unity and is attained at any $x \in \mathcal{R}(A)$ (from the range $\mathcal{R}(A)$ of the operator A).

Let us summarize.

Theorem 1. *Suppose that in measurement scheme* (1), A: $\mathcal{F} \to \mathcal{R}$ *is a given linear operator,* $f \in \mathcal{F}$ *is an a priori arbitrary element, and* ν *is a fuzzy element of* $\mathcal{R}$ *with distribution* $\varphi^\nu(\cdot)$: $\mathcal{R} \to [0,1]$; *i. e., suppose the model* $[A, \varphi^\nu(\cdot)]$ *of measurement scheme* (1) *is given. If* $\varphi^\nu(x) = \rho(\|\Sigma^{-1/2}x\|^2)$, *where* Σ: $\mathcal{R} \to \mathcal{R}$ *is a positive definite operator,* $\rho(\cdot)$: $[0,\infty) \to [0,1]$ *is a strictly monotone decreasing continuous function, and* $\rho(0) = 1$, *then any fuzzy element* (9) *of* $\mathcal{U}$ *that minimizes necessity (2) of estimation error, where the fuzzy error relation is specified by an arbitrary function* $l(\cdot,\cdot)$: $(\mathcal{U} \times \mathcal{U}) \to [0,1]$ *satisfying condition* (4), *is an N-optimal estimate of* Uf, *where* U: $\mathcal{F} \to \mathcal{U}$ *is a given linear operator. In* (9), $a(\cdot)$: $\mathcal{R} \to \mathcal{N}(A)$ *is an arbitrary function; if (and only if)* $\mathcal{N}(A) \subset \mathcal{N}(U)$, *then the only fuzzy element* (8) *is an N-optimal estimate of* Uf. *For any estimate* (9), *the necessity of estimation error is zero.*

Remark. If the quality of estimation is characterized by the possibility $L(d(\cdot)) = \sup_{x\in\mathcal{R},\ f\in\mathcal{F}} \min(\varphi^\nu(x - Af),\ l(Uf, d(x)))$ of error and the optimal strategy is determined by the condition $L(d(\cdot)) \sim \min_{d(\cdot)}$, then any strategy $d^*(\cdot)$ of (9) is in addition P-optimal if $l(u,v) = l_0(u,v) = \begin{cases} 0 & \text{at } u = v; \\ 1 & \text{at } u \neq v, \end{cases}$ where $u, v \in \mathcal{U}$, although the possibility of error is then $L(d^*(\cdot)) = \sup_{x\in\mathcal{R}} \sup_{f\in\mathcal{F}} \min(\varphi^\nu(x - Af),\ l(Uf, d^*(\cdot))) = 1$.

2. REDUCTION OF MEASUREMENT (1) IN THE PRESENCE OF *A PRIORI* INFORMATION ABOUT THE SIGNAL $f \in \mathcal{F}$

Consider the problem of reduction of measurement (1), where not only the noise ν but also the signal f is described as a fuzzy element. We denote this element by φ and rewrite (1) in the form

$$\xi = A\varphi + \nu, \tag{1*}$$

where A: $\mathcal{F} \to \mathcal{R}$ is a linear operator, $\varphi \in \mathcal{F}$ and $\nu \in \mathcal{R}$ are the independent fuzzy elements taking values in the Euclidean spaces $\mathcal{F}$ and $\mathcal{R}$, and $\pi^\varphi(\cdot)$: $\mathcal{F} \to [0,1]$ and $\pi^\nu(\cdot)$: $\mathcal{R} \to [0,1]$ are their distributions; in other words, $[A, \pi^\varphi(\cdot), \pi^\nu(\cdot)]$ is the possibility-theoretic model of scheme (1*). Let $U\varphi$ be the fuzzy vector of object parameters and U: $\mathcal{F} \to \mathcal{U}$ be a given linear operator specifying the interpretation model of the measurement ξ [1].

In the problem of measurement reduction (1*), it is required to find a strategy $d(\cdot)$: $\mathcal{R} \to \mathcal{U}$ for estimating $U\varphi$ so that the fuzzy vector $d(\xi)$ can be considered the most precise version (estimate) of the fuzzy vector $U\varphi$.

The quality of a strategy $d(\cdot)$ is characterized by the necessity of estimation error [5]

$$\begin{aligned} M(d(\cdot)) &= \neg \sup_{x\in\mathcal{R},\, f\in\mathcal{F}} \min(\pi^\nu(x - Af), \pi^\varphi(f), \neg l(Uf, d(x))) \\ &= \inf_{x\in\mathcal{R},\, f\in\mathcal{F}} \max(\neg\pi^\nu(x - Af), \neg\pi^\varphi(f), l(Uf, d(x))). \end{aligned} \tag{10}$$

Note that if $\pi^\varphi(f) = 1$ for all $f \in \mathcal{F}$, i. e., all values of the signal φ in (1*) are equally possible, then (10) takes the form $M(d(\cdot)) = \neg \sup_{x\in\mathcal{R},\ f\in\mathcal{F}} \min(\pi^\nu(x - Af),\ \neg l(Uf, d(x)))$, which coincides with expression $M(d(\cdot))$ (2) corresponding to an *a priori* arbitrary signal $f \in \mathcal{F}$.

Note also that $\pi^{\xi|\varphi}(x|f) = \varphi^\nu(x - Af)$ is the conditional possibility of the equality $\xi = x$ under the condition $\varphi = f$, i. e., $\pi^{\xi|\varphi}(\cdot|f)$ is the conditional distribution of the fuzzy element ξ under the condition $\varphi = f$. Therefore,

$$\pi^{\xi,\varphi}(x, f) = \min(\pi^{\xi|\varphi}(x|f), \pi^\varphi(f)), \quad \text{where } x \in \mathcal{R}, \quad f \in \mathcal{F}, \tag{11}$$

is the joint distribution of the input signal φ and the result of measurement ξ, ($\pi^{\xi,\varphi}(x, f)$ is the possibility of the equalities $\xi = x$ and $\varphi = f$), and (10) can be rewritten as $M(d(\cdot)) = \inf_{x\in\mathcal{R},\, f\in\mathcal{F}} \max(\neg\pi^{\xi,\varphi}(x, f), l(Uf, d(x)))$, because, according to (11), $\neg\pi^{\xi,\varphi}(x, f) = \max(\neg\pi^{\xi|\varphi}(x|f), \neg\pi^\varphi(f))$.

We call a strategy $d^*(\cdot)$ N-optimal if

$$M(d^*(\cdot)) = \min_{d(\cdot):\mathcal{R}\to\mathcal{U}} M(d(\cdot)). \tag{12}$$

This means that the strategy $d^*(\cdot)$ gives a minimum necessity of error in estimating $U\varphi$ by $d^{(*)}(\xi)$.

It is known [3] that to solve (12), it is sufficient to solve a simpler problem $M(d(x)) = \inf_{f\in\mathcal{F}} \max(\neg\pi^{\xi,\varphi}(x, f), l(Uf, d)) \sim \min_{d\in\mathcal{U}}$ for every $x \in \mathcal{R}$, because any solution $d^* = d^*(f, x)$ to this problem is also a solution to problem (12). Thus, we have to solve the minimization problem

$$\max(\neg\pi^{\xi,\varphi}(x, f), l(Uf, d)) \sim \min_{f\in\mathcal{F}, d\in\mathcal{U}}. \tag{13}$$

In what follows, we assume that $l(\cdot,\cdot)$ in (13) satisfies condition (4), i. e., the error is impossible only if $d = Uf$. We have

$$\min_{d\in\mathcal{U}} \max(\neg\pi^{\xi,\varphi}(x, f), l(Uf, d)) = \max(\neg\pi^{\xi,\varphi}(x, f), \min_{d\in\mathcal{U}} l(Uf, d)) = \neg\pi^{\xi,\varphi}(x, f),$$

where the minimum with respect to d is only attained at $d = Uf$; therefore, (13) is equivalent to the problem

$$\neg\pi^{\xi,\varphi}(x, f) \sim \min_{f\in\mathcal{F}}, \tag{14}$$

because $d^* = d^*(x) = Uf(x)$ at the point $f = f(x)$ of minimum of $\neg\pi^{\xi,\varphi}(x, f)$.

Suppose that $\neg\pi^\nu(x - Af) = \vartheta_1(\|\Sigma^{-1/2}(x - Af)\|^2)$ and $\neg\pi^\varphi(f) = \vartheta_2(\|F^{-1/2}(f - f_0)\|^2)$, where $f \in \mathcal{F}$, $x \in \mathcal{R}$, $\vartheta_1(\cdot)$ and $\vartheta_2(\cdot)$ are strictly monotone increasing functions continuously differentiable in $[0, \infty)$ and taking values in $[0, 1]$, $\vartheta_i(0) = 0$ for $i = 1, 2$, and Σ: $\mathcal{R} \to \mathcal{R}$ and F: $\mathcal{F} \to \mathcal{F}$ are positive definite operators similar to the covariance operators of the error and input signal, respectively, in the measurement model $[A, F, f_0, \Sigma]$. Under these assumptions, problem (14) becomes

$$\max(\vartheta_1(\|\Sigma^{-1/2}(x - Af)\|^2), \vartheta_2(\|F^{-1/2}(f - f_0)\|^2)) \sim \min_{f\in\mathcal{F}}, \quad x \in \mathcal{R}. \tag{15}$$

Let us introduce the notation $z = F^{-1/2}(f - f_0)$, $B = \Sigma^{-1/2}AF^{1/2}$: $\mathcal{F} \to \mathcal{R}$, and $y = \Sigma^{-1/2}x - \Sigma^{-1/2}Af_0$. Problem (15) then takes the form

$$\max(\vartheta_1(\|y - Bz\|^2), \vartheta_2(\|z\|^2)) \sim \min_{z\in\mathcal{F}}. \tag{15*}$$

To solve (15), we calculate the gradients of ϑ_1 and ϑ_2 with respect to $z \in \mathcal{F}$

$$\nabla\vartheta_1(\|y - Bz\|^2) = \vartheta_1'(\|y - Bz\|^2)B^*(Bz - y),$$

$$\nabla\vartheta_2(\|z\|^2) = \vartheta_2'(\|z\|^2)z.$$

Here, $\vartheta_1'(\|y - Bz\|^2) = d\vartheta_1(r)/dr|_{r=\|y-Bz\|^2}$ and $\vartheta_2'(\|z\|^2) = d\vartheta_2(r)/dr|_{r=\|z\|^2}$. At the point $z = z^*$ of minimum (15*), one of the following conditions must hold [7]:

either

$$\begin{aligned} &\alpha \nabla \vartheta_1(\|y - Bz^*\|^2) + (1 - \alpha) \nabla \vartheta_2(\|z^*\|^2) = 0, \\ &\vartheta_1(\|y - Bz^*\|^2) = \vartheta_2(\|z^*\|^2), \end{aligned} \tag{16}$$

for some $\alpha \in [0,1]$,

or

$$\nabla\vartheta_1(\|y - Bz^*\|^2) = 0, \quad \vartheta_1(\|y - Bz^*\|^2) > \vartheta_2(\|z^*\|^2), \tag{17}$$

or, finally,

$$\nabla\vartheta_2(\|z^*\|^2) = 0, \quad \vartheta_1(\|y - Bz^*\|^2) < \vartheta_2(\|z^*\|^2). \tag{18}$$

Consider the first case. According to the first condition in (16), $\alpha\vartheta_1' B^*(Bz^* - y) + (1-\alpha)\vartheta_2' z^* = 0$. This implies that for $\alpha \in [0,1]$, we have $z^* = z(\gamma) = (B^*B + \gamma I)^{-1}B^*(y)$, where $\gamma = (1-\alpha)\vartheta_2'/(\alpha\vartheta_1')$ ($\alpha \in (0,1)$) is obtained from the second condition in (16). For simplicity, we assume that $\vartheta_1(\cdot) = \vartheta_2(\cdot) = \vartheta(\cdot)$, where $\theta(\cdot)$: $[0,\infty) \to [0,1]$ is a strictly monotone decreasing continuously differentiable function. The second condition in (16) is then equivalent to the equality

$$\|y - Bz^*\|^2 = \|z^*\|^2. \tag{19}$$

Note that $\|y - Bz^*\|^2 = \gamma^2\|(BB^* + \gamma I)^{-1}y\|^2 = r(\gamma)$ increases monotonically with respect to $\gamma \in (0,\infty)$, $\lim\limits_{\gamma\to+0} r(\gamma) = \|I - BB^-)y\|^2$, and $\lim\limits_{\gamma\to\infty} r(\gamma) = \|y\|^2$, while $\|z^*\|^2 = \|B^*(BB^* + \gamma I)^{-1}y\|^2 = s(\gamma)$ decreases monotonically with respect to $\gamma \in (0,\infty)$, $\lim\limits_{\gamma\to+0} s(\gamma) = \|B^-y\|^2$, and $\lim\limits_{\gamma\to\infty} s(\gamma) = 0$. Therefore, (19) holds for some $\gamma \in (0,\infty)$ if and only if $\|I - BB^-)y\| < \|B^-y\|$. In this case, equation (19) has the only root $\gamma = \gamma(y) \in (0,\infty)$, and

$$z^* = z(\gamma(y)) = (B^*B + \gamma(y)I)^{-1}B^*y \tag{20}$$

specifies a stationary point (15*). Since $g(z) = \max(\vartheta(\|y - Bz\|^2), \vartheta(\|z\|^2))$ is a convex function of $z \in \mathcal{R}$, z^* determined by (19) is the sought-for point of minimum of $g(\cdot)$.

If $\|I - BB^-)y\| = \|B^-y\|$, then $\gamma = \gamma(y) = 0$ ($\alpha = 1$) and $z^* = B^-y = \lim\limits_{\gamma\to+0}(B^*B + \gamma I)^{-1}B^*y$.

Consider the second case. Any solution z^* to the equation $\nabla\vartheta(\|y - Bz\|^2) = 0$ has the form [6]

$$z^* = B^-y + b, \quad b \in \mathcal{N}(B). \tag{21}$$

Since $\|y - Bz\|^2 = \|(I - BB^-)y\|^2$ and $\|z^*\|^2 = \|B^-y\|^2 + \|b\|^2$, z^* specified by (21) satisfies conditions (17) if and only if

$$0 \le \|b\|^2 < \|(I - BB^-)y\|^2 - \|B^-y\|^2, \quad b \in \mathcal{N}(B).$$

The third case is obviously impossible. Let us formulate the obtained result.

Theorem 2. *Suppose that in measurement scheme (1*), A: $\mathcal{F} \to \mathcal{R}$ is a given linear operator and $\varphi \in \mathcal{F}$ and $\nu \in \mathcal{R}$ are independent fuzzy elements with distributions $\pi^\varphi(\cdot)$: $\mathcal{F} \to [0,1]$ and $\pi^\nu(\cdot)$: $\mathcal{R} \to [0,1]$, respectively; i. e., suppose the model $[A, \pi^\varphi(\cdot), \pi^\nu(\cdot)]$ of the measurement scheme (1*) is given. Suppose also that $\neg\pi^\varphi(f) = \vartheta(\|F^{-1/2}(f - f_0)\|^2)$ for $f \in \mathcal{F}$ and $\neg\pi^\nu(x - Af) = \vartheta(\|\Sigma^{-1/2}(x - Af)\|^2)$ for $x \in \mathcal{R}$, where $\vartheta(\cdot)$: $[0,\infty) \to [0,1]$ is a strictly monotone increasing continuously differentiable on $[0,\infty]$ function, $\vartheta(0) = 0$, and Σ: $\mathcal{R} \to \mathcal{R}$ and F: $\mathcal{F} \to \mathcal{F}$ are positive definite operators. Put $\Delta(x) = \|(I - \Sigma^{-1/2}A(\Sigma^{-1/2}A)^-)\Sigma^{-1/2}(x - Af_0)\|^2 - \|\Sigma^{-1/2}AF^{1/2})^-\Sigma^{-1/2}(x - Af_0)\|^2$ for $x \in \mathcal{R}$.*

If $\Delta(x) < 0$ at $\xi = x$, then $d^*(x) = U\hat{\varphi}$, where $\hat{\varphi} = f_0 + FA^*(AFA^* + \omega(x)\Sigma)^{-1}(x - Af_0)$ and $\omega = \omega(x)$ is a solution to the equation

$$\omega\|(BB^* + \omega I)^{-1}y\| = \|B^*(BB^* + \omega I)^{-1}y\|.$$

Here $B = \Sigma^{-1/2}AF^{1/2}$ and $y = \Sigma^{-1/2}(x - Af_0)$. In this case,

$$M(d^*(x), x) = \vartheta(\|F^{-1/2}(A^*\Sigma^{-1}A + \omega(x)F^{-1})^{-1}A^*\Sigma^{-1}(x - Af_0)\|^2).$$

If $\Delta(x) = 0$ at $\xi = x$, then $d^*(x) = U\hat{\varphi}$, where $\hat{\varphi} = f_0 + F^{1/2}(\Sigma^{-1/2}AF^{1/2})^-\Sigma^{-1/2}(\xi - Af_0)$. In this case, $M(d^*(x), x) = \vartheta(\|(\Sigma^{-1/2}AF^{1/2})^-\Sigma^{-1/2}(\xi - Af_0)\|^2)$.

If $\Delta(x) > 0$ at $\xi = x$, then $d^*(x) = U\hat{\varphi}$, where $\hat{\varphi} = f_0 + F^{1/2}(\Sigma^{-1/2}AF^{1/2})^-\Sigma^{-1/2}(x - Af_0) + a$; here, a is an arbitrary element of $\mathcal{N}(A)$ satisfying the condition $0 \le \|F^{1/2}a\|^2 < \Delta(x)$. In this case,

$$M(d^*(x), x) = \vartheta(\|(I - \Sigma^{-1/2}A(\Sigma^{-1/2}A)^-)\Sigma^{-1/2}(x - Af_0)\|^2).$$

In all the cases, the necessity of reduction error is $M(d^*(\cdot)) = \inf_{x\in\mathcal{R}} M(d^*(x), x) = 0$.

Figure 1 shows the results of measurement reduction obtained for the possibility-theoretic model $[A, \pi^{\varphi}(\cdot), \pi^{\nu}(\cdot)]$ of scheme (1*) described in Theorem 2 and the probability-theoretic model $[A, F, f_0, \Sigma]$ of scheme (1) [1].

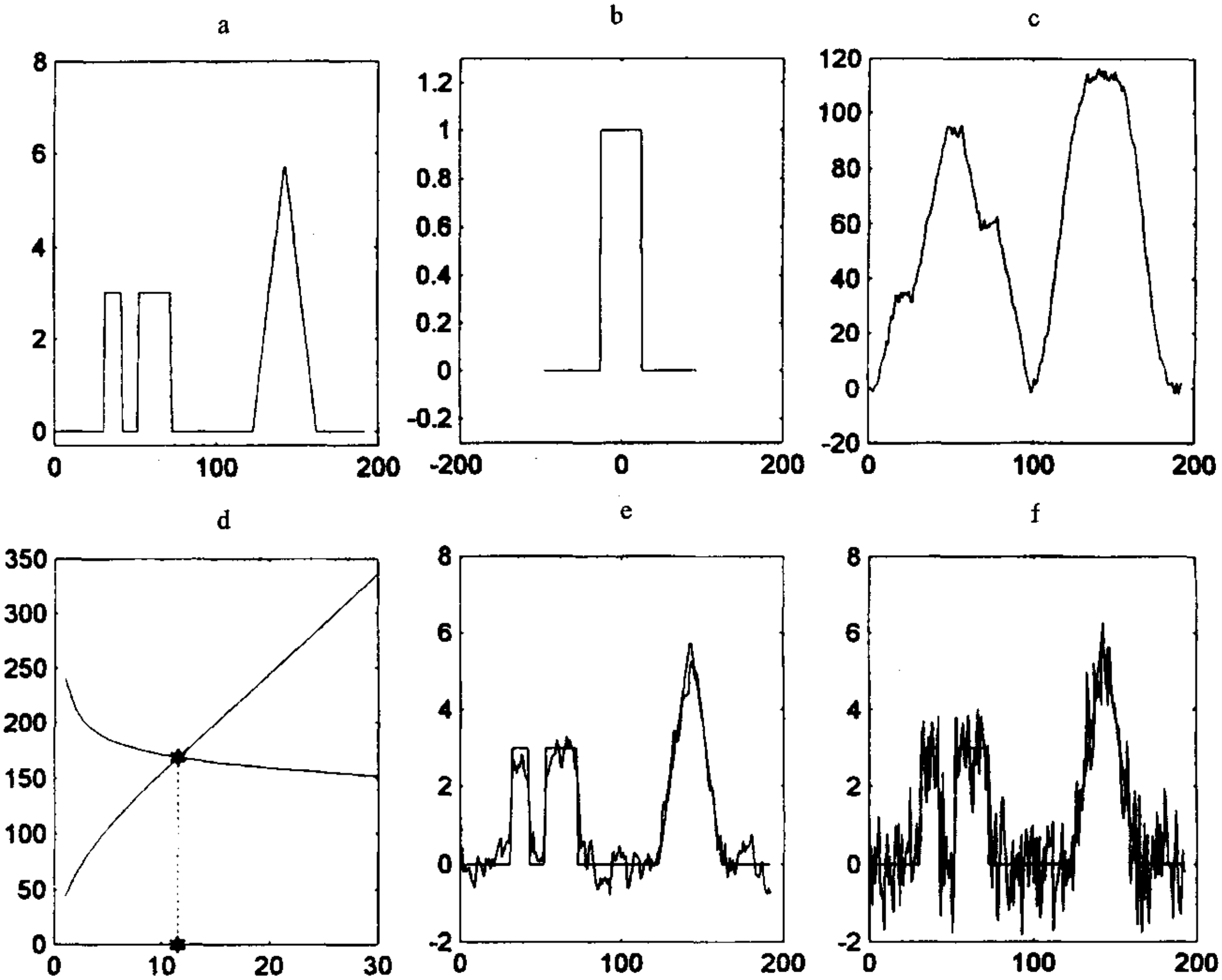

Fig. 1

Measurement reduction for the possibility-theoretic model $[A, \pi^{\varphi}(\cdot), \pi^{\nu}(\cdot)]$ and for the probability-theoretic model $[A, F, f_0, \Sigma]$: *a*—the signals f and φ measured in (1) and (1*), respectively; *b*—the instrumental function $a(i)$, where $i = -192, \ldots, 192$, of the measuring instrument A; $(Af)(i) = \sum_{j=1}^{192} a(i-j)f(j)$ for $i = 1, \ldots, 192$; *c*—measurement result $\xi(i) = (Af)(i) + \nu(i)$, where $i = 1, \ldots, 192$, in (1) and (1*); *d*—the left-hand and right-hand sides of equality (19) as functions of $\sigma^2\gamma$; the horizontal axis corresponds to the values of $\sigma^2\omega(\xi)$; *e*—the possibility-theoretic reduction of measurement (1*) $\hat{\varphi} = f_0 + FA^*(AFA^* + \omega(\xi)\Sigma)^{-1}(\xi - Af_0)$ minimizing the necessity of estimation error, $\Sigma = \sigma^2 I$, $F = \beta^2 I$, $\sigma^2 = 2$, and $\beta^2 = 2.62$; *f*—the probability-theoretic reduction of measurement (1) $\hat{f} = f_0 + FA^*(AFA^* + \Sigma)^{-1}(\xi - Af_0)$, where $\Sigma = \sigma^2 I$ and $F = \beta^2 I$ minimizing the mean-square estimation error $\mathbf{E}\|\hat{f} - f\|^2 = \min_{R:\ \mathcal{R}\to\mathcal{F}, r\in\mathcal{F}} \mathbf{E}\|R\xi + r - f\|^2$, where $\sigma^2 = 2$ and $\beta^2 = 2.62$.

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