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ON CONDITIONALLY WELL-POSED INVERSE PROBLEMS

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The possibilities of a natural conditional well-posed statement and algorithmization of two problems of immunology and seismology are discussed. The results of a mathematical experiment on estimating the modulus of continuity have been presented.

The notion of a conditionally well-posed (well-posed according to Tikhonov) problem, which was introduced in [1], generalizes Tikhonov's theorem [2] that substantiates a search method for solving inverse problems of the type of interpreting the physical experiment data. Such problems lead to the operator equation

$$Az = u, z \in Z, u \in U,$$

where Z and U are metric spaces, and are usually unstable in their initial statements, i. e., when U is mapped into Z, even if they have a unique solution with exact input data. According to [1], in the conditionally well-posed statement, a subset $\overline{Z} \subset Z$ (well-posedness set) is selected such that (i) it contains the exact and the only solution to the problem for a given $\overline{u} \in A\overline{Z}$, and (ii) sufficiently small perturbations of $\overline{u}$, which do not derive z from $\overline{Z}$, cause arbitrarily small perturbations of the solution.

In [3], methods for constructing (explicitly or algorithmically) the well-posedness set by solving an auxiliary problem with the use of quantitative or qualitative information about the sought-for solution are suggested; the algorithms for solving the auxiliary problems are regularizing in the sense of Tikhonov.

Of interest, however, are the inverse problems, where the well-posedness sets are specified even by their initial statement, i. e., in a natural way, as distinct from, e.g., compact sets that are artificially constructed in applying the classical search method. To solve these inverse problems, usual algorithms of computational mathematics are appropriate.

This paper deals with two such problems from different fields of application. Their well-posedness, according to Tikhonov, is established and, by means of a mathematical experiment, the modulus of continuity [1], i. e., the dependence of the error in a solution on the error in the input data is estimated for typical solution models. Such an estimate may be useful in planning the physical experiment providing the input data.

1. The immunity of an organism to harmful impacts is ensured by the activity of certain blood cells [4], which are characterized by an activity x. The distribution $n = n(x)$ of the number of such cells according to the activity can be described by a solution to the problem [5]

$$\frac{dn}{dx} = \frac{1}{\Phi(x)} \left(\lambda - k\dot{\Phi}(x)\right) n, \quad x > x_0, \quad n(x_0) = n_0, \tag{1}$$

where λ and k are constants and $\Phi(x)$ is a positive function parameter describing the rate of a spontaneous activity reduction.

A solution to (1) can be examined experimentally with the use of a special-purpose instrument (flow cytometer [6]), and the inverse problem is to determine the "causal" function $\Phi(x)$ from the observed effect $n = n(x)$.

In [5], this problem was solved for a narrow class of analytically specified functions Φ dependent on two numerical parameters, which were to be determined. We suggest a more general statement of the problem.

We assume that the function $\Phi(x)$ belongs to the set M of positive functions twice differentiable on $[x_0, x]$ and satisfying the condition $\Phi(x_0) = \varphi_0$ at a preset φ_0. Let $\mathcal{N}$ be the set of mappings of M by means of an operator A that gives a solution to (1)

$$A\Phi = n(x), \quad \Phi \in M. \tag{2}$$

Theorem 1. *Problem (2) is conditionally well-posed on M for any function $n \in \mathcal{N}$.*

We note first that M is a subset of the domain of the operator A, and $n(x)$ is continuously differentiable and does not vanish on M. A solution $\varphi(x)$ to problem (2) exists for any $n \in \mathcal{N}$ by the definition of $\mathcal{N}$. On the other hand, $\Phi(x)$ obviously coincides with the solution of the Cauchy problem for a first-order ordinary linear differential equation with twice differentiable coefficients. Therefore, the solution to (2) is sole. The stability of the solution on the well-posedness set M readily follows from its analytical representation

$$\Phi(x) = \frac{\lambda}{k}\int_{x_0}^{x}\left(\frac{n(x)}{n(\xi)}\right)^{1/k} d\xi + \varphi_0\left(\frac{n(x)}{n(x_0)}\right)^{1/k}. \tag{3}$$

Thus, in the problem under consideration, the well-posedness set is introduced by the very statement of the problem as a set of solutions to a certain Cauchy problem with minimal additional data.

Representation (3) shows that to find the sought-for function in practice, it is sufficient to know only $n = n(x)$. Since the latter function is experimentally determined with some nonremovable error, the study of the modulus of continuity of the inverse problem acquires a fundamental importance.

The corresponding mathematical experiment is performed for the following "sample" function suggested in [5]

$$\Phi(x) = \varphi_0\, \mathrm{e}^{\omega x}. \tag{4}$$

This function corresponds to a distribution that essentially describes the behavior of histograms of a certain type

$$\nu(x) \equiv \frac{n(x)}{n_0} = \exp\left(\frac{\lambda}{\varphi_0\omega}(1 - \mathrm{e}^{-\omega x}) - k\varphi_0 x\right). \tag{5}$$

In this experiment, the exact solution (4) of the inverse problem (2), (3) is known in advance, error (5) in the input data is controlled automatically ($\nu(x) \to \tilde{\nu}(x)$), and the corresponding perturbed value $\tilde{\Phi}(x)$ can be found by formula (3).

In the experiment, the $\nu(x)$ values were calculated on a uniform sufficiently fine grid $\{x_j\} \subset [x_0, \hat{x}]$ not to introduce computational errors for preset x_0 and $\hat{x}$; on this grid, the integral from (3) was approximated by the trapezium formula. The error in input data was modeled by the formula

$$\tilde{\nu}_j = \nu_j\left(1 + \delta\frac{\eta_j}{\sqrt{\sum\limits_j {\nu_j}^2{\eta_j}^2}}\right), \tag{6}$$

where $\{n_j\}$ are the random numbers distributed uniformly over $[-1, 1]$. The error introduced is in this case relative, and δ is its mean square measure: $\sum\limits_j(\tilde{\nu}_j - \nu_j)^2 = \delta^2$.

The approximate values of $\tilde{\Phi}$ can be obtained by formula (3) considered on a coarser (ten times in our experiment) grid $\{x_s\}$. However, at large error values $\tilde{\nu}$, pointwise approximation is impracticable, and the sought-for function was also approximated in mean square. The approximate solution was then obtained by minimizing the function

$$\Psi(\{\Phi_s\}) \equiv \sum_{\delta}\left(\Phi_s - \tilde{\Phi}_s(\tilde{\nu})\right)^2, \tag{7}$$

where $\tilde{\Phi}_s(\tilde{\nu})$ were calculated on the coarse grid according to (3). To minimize (7), the Rosenbrock method [6] was applied. Both approaches gave similar stable results in a fairly wide range of error values.

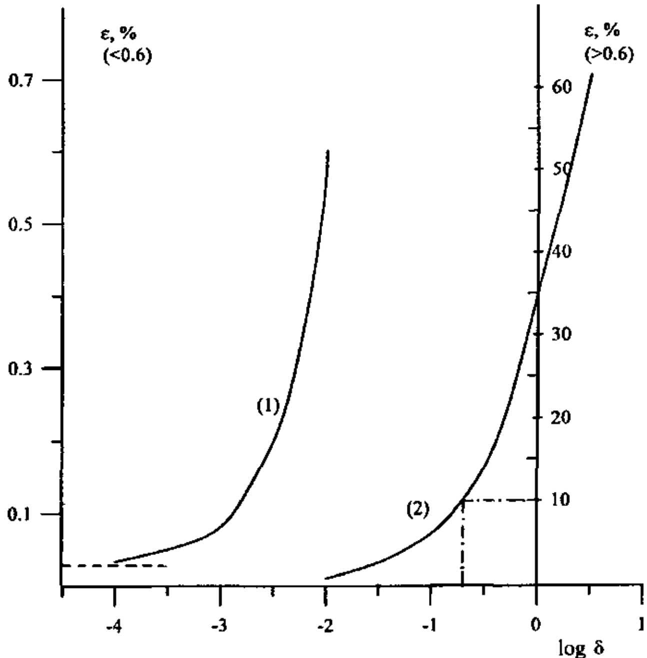

Fig. 1

Modulus of continuity for the inverse problem of immunology: $\varepsilon = \varepsilon(\delta)$ within the range of $10^{-5} \leqslant \delta \leqslant 10^{-2}$ (*1*) and $\varepsilon = \varepsilon(\delta)$ within the range of $10^{-2} \leqslant \delta \leqslant 10$ (*2*); the dashed line shows the error level of the method and the dot-and-dash line, the bound of admissible errors ($\varepsilon < 10\%$).

The obtained approximate values of $\tilde{\Phi}$ were compared to the exact values of (4), and the relative error of the result was evaluated as

$$\varepsilon = \max_s \left| \frac{\tilde{\Phi}_s - \Phi_s}{\Phi_s} \right|. \tag{8}$$

The dependence $\varepsilon = \varepsilon(\delta)$ shown in Fig. 1 refers to the following values of the parameters: $\lambda = k = \omega = \varphi_0 = 1$, $x_0 = 0$, $\hat{x} = 10$, and the number of nodes in the grid $\{x_j\}$ is 100. It is seen that the admissible experimental error, which depends on ν at each point, is of the order of about 10^{-1}. Note that an attempt to give up this dependence assuming that the error level at all points is the same

$$\tilde{\nu}_j = \nu_j + \delta \frac{\eta_j}{\sqrt{\sum_j \eta_j^2}}, \tag{9}$$

leads to a substantially worse result. The error then reaches 50% even at $\delta = 10^{-3}$. Thus, although the problem is stable, the models with a wide range of orders of the $n(x)$ values require an enhanced experimental accuracy.

2. The problem of determining the characteristics of true earth tremor from the seismograph data (the inverse problem of seismography) was for the first time considered in [7], where the exact analytical representations of the solution were applied to simple models, which made an analysis of well-posedness impossible.

Note, however, that the approach outlined in [7] includes two statements of different types, which require essentially different algorithmic treatments.

Independently of what characteristic of the input signal is to be determined, the problem related to examining the finite signal leads to the first-kind Volterra equation

$$\int_0^t K(t-\tau)x(\tau)d\tau = y(t), \tag{10}$$

where $K(t-\tau)$ is the pulse response of the instrument, $x(t)$ is the characteristic of the signal acting on the instrument starting with the moment of time $t = 0$, and $y(t)$ is the instrument reading.

In (10), $x(\tau)$ is proportional to the force acting on the movable system of the seismograph: $x(\tau) = -V_0\ddot{\xi}(\tau)$, where $\xi(\tau)$ is the true earth tremor and V_0 is some numerical parameter (magnification of the instrument [8]). The function $x(\tau)$ is of interest in its own right, and equation (10) considered relative to this function is, as is known, an ill-posed problem that requires application of regularizing algorithms.

However, of interest may be the function $\xi(\tau)$, which describes the deviation of the point of contact between the point on the earth surface and the instruments from the equilibrium position. In what follows, we show that the problem is conditionally well-posed with respect to this function and admits the application of classical computational algorithms.

Consider the set M of functions twice continuously differentiable on $[0, T]$ $(0 \leq \tau \leq t \leq T)$ and satisfying (for definiteness) the conditions $\dot{\xi}(0) = \xi(T) = 0$. Let us apply an explicit expression [8] $K(u) = (1/\omega)e^{-\sigma u}\sin\omega u$ for the kernel K for the known values of the seismograph numerical parameters (σ, u).

Integration by parts in (10) yields the following second-kind Volterra integral equation with respect to $\xi(t)$

$$\xi(t) = Y(t) + \int_0^t \varkappa(t-\tau)\xi(\tau)d\tau \equiv A\xi + Y, \tag{11}$$

where $\varkappa(t) \equiv (1/\omega)e^{-\sigma u}\left[(\sigma^2-\omega^2)\sin\omega u - 2\sigma\omega\cos\omega u\right]$ and $Y(t)$ is a continuous function determined by the readings of the instrument: $Y(t) = y(t)/V_0$.

Theorem 2. *Problem* (11) *is conditionally well-posed on the set M.*

Indeed, if $Y(t) \in \mathcal{N}$ is the set of mappings of M by means of (10), as it follows from the construction of (11), then this second-kind Volterra equation has a solution, and this solution belongs to M. In [9], such equations were considered in a more general case: for an arbitrary continuous $Y(t)$, the classical iteration process

$$\xi_{n+1} = A\xi_n + Y \tag{12}$$

was applied to establish the uniqueness of the solution. The same formula (12) implies that if $v(t) \equiv Y^{(1)}(t) - Y^{(2)}(t)$ and $\max\limits_{[0,T]}|v(t)| < \delta$, then, for $\sigma(t) \equiv \xi^{(1)}(t) - \xi^{(2)}(t)$, we have $\max\limits_{[0,T]}|\sigma(t)| \leqslant \delta e^{BT}$, where B is the upper estimate of $\varkappa(u)$. This proves the stability of the solution.

Thus, the well-posedness of the inverse seismometry problem under consideration follows from the natural reduction of this problem to a second-kind Volterra integral equation with minimal additional information on the solution sought.

Mathematical experiments on the standard function gave

$$\hat{\xi}(t) = \begin{cases} t\,e^{-\alpha t}\sin\beta t, & 0 \leq t \leq T, \\ 0, & t < 0, \quad t > T. \end{cases} \tag{13}$$

Note that equation (11) can be solved by two methods:

(a) the method of successive approximations on an approximating grid;

(b) the method of reduction to an algebraic system with subsequent application of the standard procedure (in the case under consideration, the Gauss procedure with selection of the leading term was applied).

The second method proved to be more economic for a wider range of errors.

To estimate the modulus of continuity in the mathematical experiment, an absolute error of input data was modeled by the formula

$$\tilde{y}_j = y_j + \delta\frac{\chi_j}{\sqrt{\Delta t\sum\limits_j \chi_j^2}}, \tag{14}$$

where Δt is the grid step size and $\chi_j \in [-1, 1]$ is a uniformly distributed random value, here, $\{y_j\}$ is the grid approximation of $y(t)$ computed in advance according to (13).

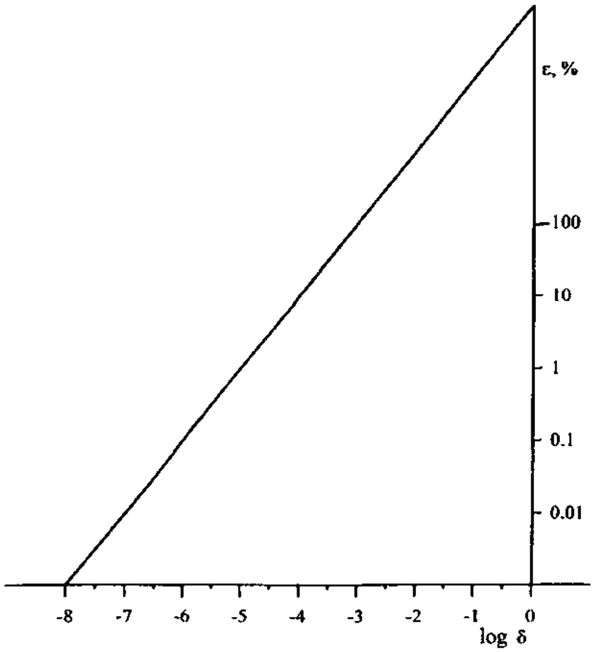

Fig. 2

Modulus of continuity for the inverse problem of seismometry.

The error of the result (which was known *a priori* in the experiment) was estimated by

$$\varepsilon = \max_i |\xi_i - \hat{\xi}_i|. \tag{15}$$

The result shown in Fig. 2 on the logarithmic scale refers to the parameter set [5] $\sigma = 0.07$; $\omega = 1.12$; $\alpha = 1.1$; $\beta = 0.7$; $V_0 = 1$; and $T = 9.6$ (s). The finite approximation was performed on a uniform grid $\{t_j\} \subset [0, T]$ of step size $\Delta t = T/50$.

It is seen that beyond the range of the method error, $\varepsilon = \varepsilon(\delta)$ is a virtually linear function, and the limit of an admissible experimental error (for $\varepsilon \leqslant 70\%$) is $\delta = 10^{-4}$.

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