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RENORMALIZATION ALONG THE LINES: NORMAL AND ANOMALOUS WARD IDENTITIES IN THE YANG–MILLS MODEL

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A generating equation for normal and anomalous Ward identities in the Yang–Mills model has been obtained in the framework of the dimensional renormalization along the lines. It has been confirmed that only one-loop diagrams are responsible for the chiral anomalies.

In paper [1], a new renormalization scheme has been proposed, i. e., renormalization along the lines. A version of dimensional regularization with only integer and positive dimensionalities employed has been presented in the framework of this scheme in [2, 3]. And finally, a generating functional has been derived in [4] for the Ward identities, so that the Green functions, renormalized according to this scheme, were subject to them. A general slightly modified procedure developed in [4] is employed for obtaining normal and anomalous Ward identities in the particular case, i. e., in the Yang–Mills model.

The basis of the renormalization procedure employed is the operation of "renormalized integration" in the momentum space (here assumed to be Euclidean). In order to define the operation at the intermediate stages uniquely, the dimensionality 2ζ of the momentum space has to be assumed high enough. However, when only a finite number of Feynman diagrams are considered, we may confine ourselves to the case of finite dimensionality. The minimal admissible dimensionality is determined by the number of the "external momenta" k_i $(i = 1, \ldots, m)$ that enter into the diagrams in question. Not only the momenta themselves, but also other vector or tensor quantities, such as vector or tensor fields, as well as "tensor currents", can play the role of these momenta (see [3]).

When the Yang–Mills model is considered, this means that the action S should be defined in a space of finite, though high enough, dimensionality, whose minimal value depends on the order of the perturbation theory, we want to confine ourselves to. Let us assume that the effective action has a form slightly different from the conventional one

$$S = \int dp\,dp'\,\delta(p+p')\left[\bar{\psi}(p')(i\hat{p}_{(\eta)} + m)\psi(p) + \frac{1}{4}F^{\alpha}_{\mu\nu}(p')F^{\alpha}_{\mu\nu}(p) - \frac{1}{2\xi}p'_{\mu}A^{\alpha}_{\mu}(p')\,p_{\nu}A^{\alpha}_{\nu}(p) - p'_{\mu}\bar{c}^{\alpha}(p')\,p_{\mu}c^{\alpha}(p)\right]$$
$$+ig\int dp\,dp'\,dq\,\delta(p+p'+q)\left[\bar{\psi}(p')\hat{A}^{\alpha}_{(\eta)}(q)t_{\alpha}\psi(p) - f_{\alpha\beta\gamma}\bar{c}^{\alpha}(p')c^{\beta}(p)p_{\mu}A^{\gamma}_{\mu}(q)\right], \tag{1}$$

where

$$[t^{\alpha}, t^{\beta}] = i\,f_{\alpha\beta\gamma}t^{\gamma}.$$

Modification of the action consists in adding a parameter η as a subscript to quantities $\hat{p}$ and $\hat{A}^{\alpha}$. This implies

$$\hat{A}_{(\eta)} = A_{\mu}\gamma^{\mu}_{(\eta)} \equiv A_{\mu}g^{\mu}_{\nu}(\eta)\gamma^{\nu},$$

where an external tensor field $g^\mu_\nu(\eta)$ has the form

$$g^\mu_\nu(\eta) = \begin{cases} 0 & \text{for } \mu \neq \nu, \\ 1 & \text{for } \mu = \nu \leq 4, \\ \eta & \text{for } \mu = \nu > 4. \end{cases}$$

The introduction of this tensor field enables us to get rid of unphysical Dirac matrices (γ^ν for $\nu > 4$) at the necessary stage by means of the limiting process $\eta \to 0$. Sometimes the momenta as functions of the parameter η will be used: $p^\mu_{(\eta)} = g^\mu_\nu(\eta)p^\nu$.

The general formula for the generating functional of renormalized Green's functions was obtained in [2]. As applied to the Yang–Mills model in question, it looks like this:

$$Z(j) = \mathcal{N}^{-1}\exp(\Delta^g)\lim_{\eta\to 0}\left[\exp(\Delta^s_{(\eta)})\exp(-W(\varphi))C(j,\varphi)|_{\varphi^s=0}\right]\Big|_{\varphi^g=0}. \tag{2}$$

Here, the fields φ consist of two types: spinor fields $\varphi^s = \{\bar\psi,\ \psi\}$ and gauge fields $\varphi^g = \{A^\alpha_\mu,\ \bar c^\alpha,\ c^\alpha\}$, a perturbative part of the action is denoted as $W(\varphi)$, and

$$C(j,\varphi) = \exp\left\{\int dp\,[\bar\psi(p)j(p) + \bar j(p)\psi(p) + \sum_u j^g_u(p)\varphi^g_u(p)]\right\}. \tag{3}$$

The operators $\Delta^s_{(\eta)}$ and Δ^g are defined by the formula

$$\Delta^s_{(\eta)} = \int d\mu(p)\,\mathcal{P}(p_{(\eta)}\to p)\frac{\delta}{\delta\psi(p)}D^s_{(\eta)}(p)\frac{\delta}{\delta\bar\psi(-p)},$$

$$\Delta^g = \int d\mu(p)\sum_{u,v}\frac{\delta}{\delta\varphi^g_u(p)}D^g_{uv}(p)\frac{\delta}{\delta\varphi^g_v(-p)},$$

where $D^s_{(\eta)}$ and D^g_{uv} are the corresponding propagators, with subscripts u, v standing for different gauge (ghost) fields. The subscript (η) in the expression for the spinor propagator signifies that the tensor field $g^\mu_\nu(\eta)$ enters into it. The operator $\mathcal{P}(p_{(\eta)}\to p)$ means that before performing regularized integration, in the integrand $p_{(\eta)}$ should be replaced by p.

The operation of regularized integration constructed in [2] is performed in the operators $\Delta^s_{(\eta)}$ and Δ^g

$$\begin{aligned}\int d\mu(p)\,F(p^2,pk_i) &= \mathcal{L}(\epsilon\downarrow 0)(-1)^{\zeta-2}(\mu^2)^{-\epsilon}\pi^{2-\zeta+\epsilon}\Gamma^{-1}(\epsilon)\\ &\times\lim_{\beta\to 0}\int d^{2\zeta}p\,(p^2+\mu^2)^{-\beta}\int_0^\infty d\omega^2\,(\omega^2)^{\epsilon-1}\left(\frac{\partial}{\partial\omega^2}\right)^{\zeta-2}F(p^2+\omega^2,pk_i).\end{aligned} \tag{4}$$

Special features of applying this operation in the presence of vector and spinor fields have been considered in [3]. The operator $\mathcal{L}(\epsilon\downarrow 0)$ in (4) makes all the expansion terms in the Laurent series ϵ vanish in terms of ϵ with the exception of the term of order ϵ^0.

It is assumed that the above definitions of the operations $\mathcal{P}(p_{(\eta)}\to p)$ and of renormalized integration are valid only when p is a loop momentum. In this case, the renormalized integration effectively removes ultraviolet divergences (see [2]). If momentum p propagates along an open line, then $\mathcal{P}(p_{(\eta)}\to p)$ should be considered as a unit operator, and the renormalized integration as coincident with the ordinary integration.

The renormalized integration shows translational invariance

$$\int d\mu(p)\,F(p+q;k_1,\ldots) = \int d\mu(p)\,F(p;k_1,\ldots). \tag{5}$$

This property follows from the fact, proved in [2], that renormalized integration is equivalent to the dimensional regularization in Wilson's version [5], where this property takes place (see [6]).

Returning to action (1), one can easily verify that it is invariant under the ordinary BRST-transformations [7, 8]

$$\begin{aligned}
\delta\psi(p) &= i\,g\,t_\alpha \int dq\, c^\alpha(p-q)\psi(q)\delta\lambda, \\
\delta\bar{\psi}(p) &= i\,g \int dq\, \bar{\psi}(q) t_\alpha c^\alpha(p-q)\delta\lambda, \\
\delta A_\mu^\alpha(p) &= ip_\mu c^\alpha(p) + g\, f_{\alpha\beta\gamma} \int dq\, c^\beta(p-q) A_\mu^\gamma(q)\delta\lambda, \\
\delta c^\alpha(p) &= -\frac{1}{2} g\, f_{\alpha\beta\gamma} \int dq\, c^\beta(p-q) c^\gamma(q)\delta\lambda, \\
\delta\bar{c}^\alpha(p) &= -\frac{i}{\xi} p_\mu A_\mu^\alpha \delta\lambda.
\end{aligned} \tag{6}$$

Introducing a unified notation for field variations

$$\delta\varphi_v^n(p) = f_v^n(p,\varphi)\delta\lambda \qquad (n = g, s), \tag{7}$$

one can write the action variation in the form

$$\delta S(\varphi) = \int dp \sum_{n,v} \frac{\delta S(\varphi)}{\delta\varphi_v^n(p)} f_v^n(p,\varphi)\delta\lambda. \tag{8}$$

Derivation of the generating equation for the Ward identities has been given in [4], where a possibility of introducing an additional tensor field $g_\nu^\mu(\eta)$ into the action and passage to the limit $\eta \to 0$ has not been considered, though. However, the conclusion drawn in that paper is applied to our case practically without any alterations. As a result, the following formula for the generating equation is obtained

$$\mathcal{N}^{-1} \int d\mu(p)\, \exp(\Delta^g) \lim_{\eta\to 0} \exp(\Delta^s_{(\eta)}) \exp(-W(\varphi)) C(j,\varphi) \sum_{n,v} \left[j_v^n(p) - \frac{\delta W_1(\varphi)}{\delta\varphi_v^n(p)} \right] f_v^n(p,\varphi) \Big|_{\varphi=0} = R, \tag{9}$$

where

$$R = \mathcal{N}^{-1} \int d\mu(p) \exp(\Delta^g) \lim_{\eta\to 0} \exp(\Delta^s_{(\eta)}) \exp\{-W(\varphi)\}\; C(j,\varphi) \sum_{n,v} \frac{\delta S_0(\varphi)}{\delta\varphi_v^n(p)} f_v^n(p,\varphi) \Big|_{\varphi=0}. \tag{10}$$

Here $S_0(\varphi)$ is the part of the action invariant with respect to transformations of classical fields. Accordingly, we have in (9) $W_1(\varphi) = S(\varphi) - S_0(\varphi)$. In the case of the BRST-transformations (6) we have $S(\varphi) = S_0(\varphi)$ and $W_1(\varphi) = 0$.

Compare the right-hand sides of formulas (8) and (10). The main difference between them is that variation of the action $S_0(\varphi)$ in (10) is subjected to repeated actions of the operators Δ^g and $\Delta^s_{(\eta)}$. These operators replace classical fields φ in $S_0(\varphi)$ by the corresponding propagators, whose behavior at large momenta is markedly worse than that of the fields φ. However, the ordinary integration $\int dp$ in (10) is simultaneously replaced by the renormalized integration $\int d\mu(p)$. Moreover, the renormalized integration operations defined on the propagator products are also involved in the operators Δ^g and $\Delta^s_{(\eta)}$.

The fact that the variation of the action $S_0(\varphi)$ vanishes under transformations (6) is a consequence of the translational invariance of the ordinary integration $\int dp$ in formula (8). Since the renormalized integration operation in (10) also demonstrates translational invariance, the right-hand side of (10) vanishes as well.

Thus, one has to put $W_1 = 0$ and $R = 0$ in (9). Then it transforms into a generating equation for the Ward identities for the renormalized Green's functions. In order for this equation to be interpreted as a certain equation for the generating functional Z, sources j_v^n of composite fields φ_v^n have to be introduced in addition to sources J_v^n of elementary fields ϕ_v^n. The reason is that transformations (6) are nonlinear with respect to fields φ. The quadratic combinations of the fields appear in them. These combinations should be considered as composite fields ϕ_v^n and formula (7) should be rewritten as

$$\delta\varphi_v^n(p) = [f_{1v}^n(p;\varphi) + f_{2v}^n(p;\phi)]\,\delta\lambda,$$

where $f_{1v}^n(p;\varphi)$ and $f_{2v}^n(p;\phi)$ are nonlinear with respect to φ and ϕ respectively.

After this one has to introduce the generating functional $Z(j,J)$ defined in (2), where $C(j,\varphi)$ is replaced by $C(j,\varphi;J,\phi)$. In its turn, $C(j,\varphi;J,\phi)$ is defined by (3), where $\sum_{n,v} J_v^n(p)\phi_v^n(p)$ is added to the integrand.

One may always assume the sources $j_v^n(p)$ and $J_v^n(p)$ to be functions with satisfactory enough properties and the momentum p (in the Feynman diagram) to be produced by one of these sources. Since the renormalized integration for nonloop momenta coincides with the ordinary one, equation (9) (for $W_1 = 0$ and $R = 0$) may take the form

$$\int dp \sum_{n,v} j_v^n(p)\left[f_{1v}^n(p;\frac{\delta}{\delta j}) + f_{2v}^n(p;\frac{\delta}{\delta J})\right] Z(j,J)\Big|_{J=0} = 0.$$

Consider now the local chiral transformation

$$\begin{aligned}\delta\psi(p) &= g\int dq\,\gamma_5\psi(p-q)\delta\lambda(q),\\ \delta\bar\psi(p) &= g\int dq\,\bar\psi(p-q)\gamma_5\delta\lambda(q),\end{aligned} \tag{11}$$

which at the classical level is related to the (partial) axial current conservation. At the quantum level, however, even in electrodynamics, it leads to the Adler anomaly [9]. The anomalies are explained by the presence of γ_5-matrices in transformation (11).

In the 4-dimensional space the γ_5-matrix is defined by means of the totally antisymmetric tensor $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$. Following [10], we consider it as a constant external field. Since a space of greater dimensionality 2ζ has to be employed in the renormalized integration, the tensor $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ should be extended to the case of this space. The extension may be realized by various means, which reduces to finite renormalization. We take the simplest variant. A constant external totally antisymmetric fourth-order tensor field with only physical components being nonvanishing, and with the component E_{1234} being equal to unity can be used as an extension. The matrix γ_5 can be defined by this field in the following way:

$$\gamma_5 = \frac{1}{4!}E_{\mu_1\mu_2\mu_3\mu_4}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}.$$

It is obvious that γ_5 anticommutes with matrices γ^μ at $\mu = 1,2,3,4$ and commutes with them at $\mu > 4$.

The term in action (1) proportional to the spinor mass violates chiral invariance already at the classical level. This is manifested by the fact that the following term appears in the right-hand side of (9)

$$\sum_{n,v}\frac{\delta W_1(\varphi)}{\delta\varphi_v^n(p)}f_v^n(p,\varphi) = -2mg\,\bar\psi(q-p)\gamma_5\psi(p).$$

Quantum anomalies will arise together with this, as in this case the quantity R turns out to be nonzero. Since only spinor fields take part in transformations (11), the renormalized integration $\int d\mu(p)$ in (9) and (10) can be carried out right after the operator $\exp(\Delta^s_{(\eta)})$ action. Hence, in this case the expression for R is as follows:

$$R = \mathcal{N}^{-1}\exp(\Delta^g)\lim_{\eta\to 0} Q(\varphi^g, j^s)C(j^g,\varphi^g)|_{\varphi^g=0}, \tag{12}$$

where

$$\begin{aligned}Q = \int d\mu(p)\,\exp(\Delta^s_{(\eta)})\exp\{-W(\varphi)\}C(j^s,\varphi^s)\,ig\,\bar\psi(q-p)\{\gamma_5,\gamma^\nu_{(\eta)}\}\\ \times\left[p_\nu\psi(p) + g\int dk\,A_\nu^\alpha(k)\,t_\alpha\psi(p-k)\right]_{\varphi^s=0}.\end{aligned} \tag{13}$$

The expression for Q can be depicted graphically in the form of a collection of Feynman diagrams that consist of closed spinor loops and open spinor tracks. The latter connect the spinor currents $\bar{\jmath}^s$ and j^s to each other. Moreover, there are external lines that correspond to gauge fields φ^g. There are no internal lines of this kind, and hence both spinor loops and spinor tracks are not connected.

The Dirac matrix $\gamma^{\nu}_{(\eta)}$ in the right-hand side of (13) may appear either in one of the loops or in a track. In the second case, the anticommutator $\{\gamma_5, \gamma^{\nu}_{(\eta)}\}$ vanishes on passing to the limit $\eta \to 0$, and the corresponding term from Q does not contribute to R. The same situation takes place when the matrix $\gamma^{\nu}_{(\eta)}$ belonging to the loop is contracted to the field $A^{\alpha}_{\nu}(k)$ (the second term in the square brackets in the right-hand side of (13)).

When $\gamma^{\nu}_{(\eta)}$ belongs to the loop and is contracted to p_ν (the first term in the square brackets in (13)), the situation is different. The terms proportional to p^2 appear in this case upon the calculation of traces and upon the replacement $p_{(\eta)} \to p$. According to (4), ω^2 should be added to p^2 in calculating the renormalized integral. If the integral over p in (4) converges at $\beta = 0$, then the operation $\mathcal{L}(\epsilon \downarrow 0)\,\Gamma^{-1}(\varepsilon) \int d\omega^2\,(\omega^2)^{\epsilon-1}\ldots$ is equivalent to the operation $\int d\omega^2\,\delta(\omega^2)\ldots$ (see [11]). Correspondingly, the term proportional to ω^2 does not contribute to Q and there will be no anomalies. However, if the above integral diverges, the contribution to Q is finite even at $\eta \to 0$.

This contribution is the source of anomalies. In the model in question, unlike electrodynamics, contribution to anomalies is provided not only by loops with three links but also by those with five links. Only these one-loop diagrams contribute to the anomalies.

As there is a limiting process $\eta \to 0$ in (12), no other chiral anomalies can arise, since only physical γ-matrices "survive" after this passages to the limit.

The latter has one more consequence, which has to be taken into account in calculating further renormalized integrals corresponding to internal gauge lines. The thing is that the integrand can be a function both of k^2 and of $\mathbf{k}^2 = k_\mu k_\nu g^\mu_\nu$ $(\eta = 0)$. The quantity k^2 should be considered as a contraction of the integration momentum to an external tensor. Therefore, in calculating a renormalized integral, one should add ω^2 only to k^2 rather than to $\mathbf{k}^2$.

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