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SURPRISES OF QUANTUM MEASUREMENTS

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Formal analysis of the state of an object after measuring its coordinate shows that the conditional variance of the momentum may depend on the result of coordinate measurement and be smaller than the momentum variance in the initial state even if there is no correlation between the momentum and the momentum squared with the coordinate. This fact cannot be interpreted on the basis of ordinary statistics, exemplifying still another nonclassical consequence of the quantum theory.

Let us discuss possible results of the following experiment. In a given ensemble, we first approximately measure the coordinate x of each particle, and then we measure precisely the conjugate momentum p. The coordinate is measured indirectly, i. e., the evolution of particles remains the same both for pre- and post-measurement. This experiment yields a set of number pairs $(\tilde{x}_i, \tilde{p}_i)$, the results of consecutive coordinate and momentum measurements. Let us select a subset of pairs $(\tilde{x}_i, \tilde{p}_i)$ with a specified value $\tilde{x}$, and calculate the momentum distribution density $w(\tilde{p}|\tilde{x})$ that corresponds to this subensemble. From the standpoint of quantum measurement theory, this *a posteriori* momentum distribution density will be the momentum distribution density in the particle state after the coordinate measurement, which yielded the result $\tilde{x}$.

A mathematical apparatus developed in measurement theory relates the *a posteriori* state of a particle with its initial state and with the accuracy of its coordinate measurement. Such an analysis will be carried out in what follows. But first we try to picture its results based on certain assertions, which appear to be doubtless.

1. On measuring the coordinate, the device supplies a random momentum to the particle (e.g., on photon scattering), which is added to its initial momentum. Therefore the momentum distribution density calculated in pairs $(\tilde{x}_i, \tilde{p}_i)$ (the unconditional distribution density) will correspond to the distribution density of a sum of two independent random variables. The unconditional variance of the momentum will be

$$\Delta^2\tilde{p} = \Delta_0^2 p + \Delta_p^2 p,$$

where $\Delta_0^2 p$ is the variance of p in the initial state of the particle and $\Delta_p^2 p$ is the variance of momentum perturbation by the device.

2. Can the conditional momentum variance $\Delta^2\tilde{p}(\tilde{x})$ be greater or smaller than the unconditional one? Can it be greater or smaller than the variance in the initial state? Evidently, it can if in the initial state the particle momentum correlates with its coordinate, because in that case the sorting of measurement results with respect to $\tilde{x}$ will be accompanied by their sorting with respect to $\tilde{p}$. But what if there is no such a correlation and, in addition, the momentum perturbation does not correlate with the result of the coordinate measurement? In that case, from the standpoint of classical physics it is impossible to explain why the conditional momentum variance differs from the unconditional one, or especially why it turns out to be smaller than the variance in the initial state.

The formal analysis in the framework of the quantum theory given below yields a different answer. The conditional momentum variance may depend on the coordinate measurement result and be smaller

than the momentum variance in the initial state even in the absence of the above-mentioned correlations. This is an example of still another nonclassical consequence of the quantum theory.

Fundamental principles of the quantum theory of measurements were formulated by the founders of quantum mechanics [1–3] and discussed in many publications. A deep analysis of the theory of indirect measurements was carried out by L. I. Mandel'shtam [4]. The modern quantum theory of measurements differs from its original form mainly by a more advanced body of mathematics.

The process of indirect measurement is pictured as follows [4–6]. At first, the quantum part of the device (the quantum reading system, or QRS) interacts with the object. At this stage an ordinary dynamic interaction of two quantum systems takes place. Then the QRS interacts irreversibly with the device classical part, which results in conveying—already on the classical level—information about the importance of a certain observable QRS. To use this measurement result for evaluating an observable $\hat{A}$ of the object, the observable QRS must depend on $\hat{A}$ after the interaction of the device with the object. To do this, it will suffice that the Hamiltonian of the QRS–object interaction be equal to

$$\hat{H}_i = -\alpha_0 \hat{A}\hat{Y},$$

where $\hat{Y}$ is some QRS operator and α_0 is a c-numerical coefficient, which only remains other than zero over a short period of time τ. This condition, dated back to von Neumann [1], is referred to as the standard measurement scheme.

If before the interaction the object was in a state described by the density matrix $\hat{\rho}_0$ and the QRS was in a state $\hat{\rho}_a$, then at the moment of interaction break their common state will be

$$\hat{\rho}(\tau) = \hat{U}(\tau)\hat{\rho}_0\hat{\rho}_a\hat{U}^+(\tau),$$

where $\hat{U}(\tau)$ is the operator of the joint evolution of the object and the device.

If the quantities of α_0 and τ satisfy the condition of instant measurement, i. e., $\alpha_0\tau = \alpha$, $\alpha_0\tau^2 \to 0$, then we may assume that

$$\hat{U}(\tau) = \exp\{i\alpha\hat{A}\hat{Y}\}.$$

Here and in what follows we use the system of units, in which $\hbar = 1$. The state $\hat{\rho}(\tau)$ differs from the initial one in that the observable conjugate to $\hat{A}$ is shifted by $\alpha\hat{Y}$, and the QRS observable $\hat{P}$ conjugate to $\hat{Y}$ is shifted by $\alpha\hat{A}$.

After the QRS–object interaction, the measurement of P occurs, which yields some value $\tilde{P}$. The quantity $(\tilde{P} - \langle P(0)\rangle)/\alpha$ is taken as the result of measurement $\tilde{\alpha}$ of the object observable $\hat{A}$. It corresponds to the maximum likelihood estimate. (Henceforth we will assume $\langle P(0)\rangle = 0$.)

The conditional distribution density of the object observable B is

$$w(B|\tilde{P}) = \frac{\langle B|\langle\tilde{P}|\hat{\rho}(\tau)|\tilde{P}\rangle|B\rangle}{w(\tilde{P})}, \tag{1}$$

where the normalizing factor $w(\tilde{P})$ is the unconditional distribution density of $\tilde{P}$.

We carry out further calculations for the case of pure initial states of the object and the device, i. e., for $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$, $\hat{\rho}_a = |\varphi_a\rangle\langle\varphi_a|$. Under these conditions, simple transformations yield

$$w(B|\tilde{P}) = \frac{\left|\int \psi_0(A)\langle B|A\rangle\langle\tilde{P}|\,e^{i\alpha A\hat{Y}}\,|\varphi_a\rangle dA\right|^2}{w(\tilde{P})}. \tag{2}$$

This corresponds to the distribution density of B in the pure state

$$|\psi_1\rangle = \frac{\int \psi_0(A)\varphi_a(\tilde{P} - \alpha A)|A\rangle dA}{w(\tilde{P})^{1/2}}. \tag{3}$$

In particular, if A is the coordinate and B is the momentum, then the wave function corresponding to (3) is

$$\psi_1(x|\tilde{x}) = \frac{\psi_0(x)\varphi_a(\tilde{P} - \alpha x)}{w(\tilde{x})^{1/2}}.$$

In the general case, the shape of the wave function ψ_1 depends on the shape of ψ_0 and φ_a and on the value of $\tilde{x}$. Only in some particular cases will the shape of ψ_1 be similar to that of ψ_0 — for instance, when ψ_0 and φ_a are Gaussian. Since the wave function shape depends on $\tilde{x}$, the *a posteriori* distribution density of momentum will depend on the result of the coordinate measurement.

Assume that

$$\varphi_a(\tilde{P}-\alpha x)=\frac{\exp\{-(\tilde{P}-\alpha x)^2/(4\Delta^2 P)\}}{(\sqrt{2\pi}\Delta P)^{1/2}}=\frac{\exp\{-(\tilde{x}-x)^2/(4\Delta_m^2 x)\}}{(\sqrt{2\pi}\Delta_m x)^{1/2}},$$

where $\tilde{x}=\tilde{P}/\alpha$, $\Delta_m x=\Delta P/\alpha$, and $\psi_0(x)$ is "nearly rectangular", i. e., $\psi_0(x)\cong 1/\sqrt{l}$ for $-l/2<x<l/2$ falling rapidly (exponentially) to zero beyond this interval.

Note that in such initial states, the coordinate and the momentum do not correlate with each other, both for the object and for the device. However, the shape of the *a posteriori* wave function $\psi_1(x)$ and, respectively, the shape of the *a posteriori* distribution density of momentum depend on $\tilde{x}$. For example, for $\Delta_m x\ll l/2$ and $|\tilde{x}|\ll l/2-\Delta_m x$ the function $\psi_1(x|\tilde{x})$ will be nearly Gaussian. If the second inequality does not hold, then ψ_1 will be a Gaussian curve truncated at $|x|=l/2$. As a consequence, the *a posteriori* variance of the particle momentum in the second case will be greater than in the first one. Meanwhile, the *a posteriori* average momentum does not depend on $\tilde{x}$. This example undoubtedly proves that the conditional variance of the particle momentum may depend on the result of the coordinate measurement and respectively be greater or smaller than the unconditional variance, even if the momentum and the coordinate do not correlate in the initial state. And this is the case when the dynamic influence of the device upon the particle consists in shifting its momentum by αY independently of $\tilde{x}$.

All the more unexpected is the fact that, the initial states being the same, the conditional variance of momentum may be smaller than the momentum variance in the initial unperturbed state. And this is true despite the fact that the device increases the uncertainty of the particle momentum. In order to prove this, we have to resort to numerical calculation. The calculation of the momentum variance for a given wave function $\psi_1(x)$ requires the calculation of a double integral in a wide range of x and p values, which necessitates a large body of computer calculations. Therefore, without a decrease of proof generality, we consider an example of phase variance variation in measuring the energy of a harmonic oscillator.

We used the Pegg–Barnett operator [7] as a Hermitian phase operator, whose eigenvalue spectrum lies in the 2π interval. The oscillator initial state was the "rectangular" state

$$|\psi_0\rangle=(1/(m-k+1))^{1/2}\sum_k^m|n\rangle.$$

The phase distribution density

$$w(\phi)=\left|\sum_0^\infty \mathrm{e}^{-i\phi n}\psi(n)\right|^2$$

is in this state

$$w_0(\phi)=\frac{\sin^2((m-k+1)\phi/2)}{2\pi(m-k+1)\sin^2(\phi/2)}. \tag{4}$$

Let

$$\psi_a(\tilde{P}-\alpha n)=\frac{\exp\{-(\tilde{n}-n)^2/(4\Delta_m^2 n)\}}{(\sqrt{2\pi}\Delta_m n)^{1/2}},$$

where $\tilde{n}=\tilde{P}/\alpha$ and $\Delta_m n=\Delta P/\alpha$. We obtain

$$\psi_1(n|\tilde{n})=(1/(m-k+1))^{1/2}\sum_k^m\psi_a(\tilde{n}-n).$$

The corresponding *a posteriori* phase distribution density is

$$w(\phi|\tilde{n}) = \frac{\left|\sum_{k}^{m} \exp\{-i\phi n\} \exp\{(\tilde{n}-n)^2/(4\Delta_m^2 n)\}\right|^2}{2\pi \sum_{k}^{m} \exp\{-(\tilde{n}-n)^2/(2\Delta_m^2 n)\}}.$$

The function $w(\phi|\tilde{n})$, just like $w_0(\phi)$, is symmetrical with respect to $\varphi = 0$. Consequently, the *a posteriori* average of the $\langle\phi(\tilde{n})\rangle$ phase equals the initial average regardless of the value of $\tilde{n}$. (It is zero if the phase variation interval from $-\pi$ to π is selected.)

Calculations showed that both $w(\phi|\tilde{n})$ and $w_0(\phi)$ do not depend on the absolute values of m and k. The following results were obtained for $m - k = 5$. If $\tilde{n}$ falls into the middle of the $m - k$ interval, then the conditional phase density $\Delta_1^2\phi$ is smaller than the initial value $\Delta_0^2\phi = 0.46$ for any values of $\Delta_m^2 n > 1/(4\Delta_0^2\phi)$. The smallest value $\Delta_1^2\phi \approx 0.24$ was obtained for $\Delta_m^2 n \cong 1.5$. For the same value $\Delta_m^2 n = 1.5$, but with $\tilde{n} = 0$, it was found that $\Delta_1^2\phi = 0.82 > \Delta_0^2\phi$. The variance of dynamic phase perturbation in all cases was $1/(4\Delta_m^2 n)$. The calculation data prove that the *a posteriori* phase variance can be smaller than the phase variance in the initial state even in the absence of correlation between ϕ and ϕ^2 and n. These data also give grounds to state that a similar relationship may hold between the *a posteriori* and the initial variance of the particle momentum when measuring the coordinate.

At first glance, based on these results, the following conclusion may be drawn. Although the initial wave function does not show explicitly any dependence of the momentum on the coordinate, actually the highest momentum variance is displayed by the particles, whose coordinates are close to the jump of $\psi_0(x)$. Using the "rectangular" wave function formed with the aid of a slot as an example, this could be explained as follows. The momentum of only those particles is perturbed, which pass near the slot edge. However, this simple quasiclassical interpretation does not agree, for example, with the fact that the location of zeros in the momentum (phase) distribution density depends on the slot width (see (4)).

From the standpoint of the quantum theory, the eigenvalues of the momentum operator are the wave numbers of the waves e^{ipx}, whose superposition forms the wave function

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int \varphi(p) e^{ipx}\, dp.$$

The spectral function $\varphi(p)$ depends on the shape of the entire function $\psi(x)$, rather than on its separate intervals. A device that measures the momentum precisely yields the wave number of one of the spatial harmonics. (Such a device must have an infinite interaction domain.) Under this approach, the wave function appears to be not a mathematical abstraction but some physical reality. If the calculation of the measurement process shows that the momentum distribution density changes, then we have to agree that the wave function has changed. Then one has to accept that during the particle interaction with the QRS, there occurs not only a dynamic interaction described by the Schrödinger equation, but also a mysterious quantum process of state reduction manifesting itself not in the classical transition from *a priori* to *a posteriori* probability but in changes of the wave function shape with appropriate changes in its spectral function $\varphi(p)$. As a result of interaction with the coordinate measuring device, the wave function changes as if it passes through a spatial filter with transmission coefficient $\varphi_a(\tilde{x} - x)$, though it is not evident from the character of dynamic interaction. These arguments, formally being in agreement with mathematical reasoning, cause consequences with way out hidden from view. Therefore they cannot be regarded as unconditional description of physical reality, just like various interpretations of quantum mechanics (statistical, Copenhagen, latent-parameters, multiuniverse are not accepted as indisputable. Nine different interpretations of quantum formalism are discussed in Sudbury's book [8].)

The problem of state variation during measurement is discussed since the first basic works on quantum mechanics appeared. However, the examples considered yielded the results, which allowed one to represent the wave function reduction as a trivial transition from *a priori* to *a posteriori* coordinate distribution. If, for instance, we were interested in the *a posteriori* distribution of the coordinate $w(x|\tilde{x})$ rather than in the

momentum $w(p|\tilde{x})$, we would obtain a well-known classical Bayes's formula [4]

$$w(x|\tilde{x}) = \frac{w_0(x)w_a(\tilde{x}|x)}{w(\tilde{x})}.$$

The same formula could be obtained assuming the particle and the coordinate measurement to be classical and characterizing the state of the particle and of the device only by distribution densities rather than by wave functions. The fact that the *a posteriori* coordinate distribution is narrower than the original one could be given a simple statistical explanation. This removed the necessity of discussing the mysterious process of wave function collapse.

The properties of the *a posteriori* momentum distribution density, derived above and having no classical interpretation, as far as the author is informed, have not been tested in any experiments.

Such experiments could be based on the elegant atomic-beam technique described in [9]. It allowed one to quite reliably reconstruct the Wigner function of a beam of helium atoms that has passed through two slots. If such a device could be supplemented by some kind of indirect measurement of transverse coordinate of atoms behind the slot, the effect of wave function reduction during measurement described above could be tested.

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