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DIRAC FERMION IN STRONG COULOMB FIELD IN 2+1 DIMENSIONS

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The effect of charged fermion pair production by a strong external Coulomb field in two spatial dimensions is studied. Exact solutions to the Dirac equation have been found for a Coulomb field in 2+1 dimensions. It is shown that the behavior of the lower energy levels of an electron in a strong Coulomb field differs significantly in the cases of two and three spatial dimensions. An equation has been obtained for the critical charge and solved numerically for a simple model. The critical charge in 2+1 dimensions is much smaller than its value for the same model in 3+1 dimensions.

Considerable recent attention has been focused on two-dimensional systems of nonrelativistic charged fermions associated with the electromagnetic and Chern–Simons calibration fields owing to their rather unusual properties, which permit the use of these models to study such quantum macroscopic phenomena as the fractional quantum Hall effect and high-temperature superconductivity [1, 2]. Certain effects in condensed media physics indicate that not only (2+1)-dimensional nonrelativistic fermion systems, but also (2+1)-dimensional systems whose energy spectrum is determined by the Hamiltonian of the Dirac equation may actually exist [3–5]. The properties of quantum (2+1)-dimensional systems are also of purely theoretical interest in connection with the theory of anions, particles that are governed by a fractional statistics in 2+1 dimensions [2]. This is the motivation of the present study.

We are going to find the exact solutions to the Dirac equation for an electron in a Coulomb field, assuming its motion to be restricted to a plane, and to discuss the charged fermion pair production (i. e., electrons and positrons) from a vacuum by a strong Coulomb field in 2+1 dimensions predicted in [6] and thoroughly investigated in [7–13] for the case of 3+1 dimensions.

It is worthwhile to recall that in the (3+1)-dimensional quantum mechanics the expression for the energy of the ground state of an electron in the Coulomb field of a point charge $Z|e|$ becomes meaningless when $E_0(Z)$ vanishes. In that case, in order to find the energy spectrum of the electron in the Coulomb field, one has to set a boundary condition at $r = 0$, i. e., to consider a potential truncated at some distance R [13]. From the standpoint of physics, the use of a truncated Coulomb potential is equivalent to taking into account the finite dimensions of the nucleus that creates the potential.

In the case of three spatial dimensions, the energy spectrum of an electron in a strong field of a truncated (at small distances) Coulomb potential was first explored in [13]. It turned out that with increasing Z over the region $z > 137$, the electron energy levels become negative and keep sinking to the lower continuum boundary, $-m$. The value $Z = Z_{\rm cr}$ at which the electron lowest energy reaches the lower continuum boundary is referred to as the critical value for the ground state [10–12]. If $Z > Z_{\rm cr}$, the ground electron energy level is "immersed" into the lower continuum, and, if this level was not filled, then the arising quasistationary state results in the production of two positrons, which move to infinity by the Coulomb repulsion, while the vacuum of quantum electrodynamics (QED) perturbed by the supercritical Coulomb field, receives the charge $2e$ [10–12]. We use the system of units, in which $c = \hbar = 1$.

The quantum-mechanical problem on the motion of a relativistic electron with mass m and charge $e = -e_0$, $e_0 > 0$, in two spatial dimensions in an external Coulomb field that simulates the field of a heavy nucleus, can be solved exactly. Let us specify the vector potential of the Coulomb field in Cartesian coordinates as follows:

$$A^0(r) = -Ze_0/r, \quad A^x = A^y = 0. \tag{1}$$

Since (see, e.g., [3]) in 2+1 dimensions the algebra of Dirac's matrices can be represented in terms of the Pauli matrices, we select the representation $\gamma^0 = \sigma^3$, $\gamma^k = i\sigma^k$ and write the Dirac equation in the form

$$(i\partial_t - H_D)\Psi = 0, \tag{2}$$

where

$$H_D = \alpha\mathbf{P} + \beta m + eA^0 \equiv \sigma^1 P_2 - \sigma^2 P_1 + \sigma^3 m + eA^0, \tag{3}$$

and $P_\mu = i\partial_\mu - eA_\mu$ is the operator of the electron generalized momentum.

We seek the solution to (2) in the field (1) in polar coordinates r, φ in the form

$$\Psi(t, \mathrm{x}) = \frac{1}{\sqrt{2\pi}} \exp(-i\epsilon Et + il\varphi)\psi(r, \varphi), \tag{4}$$

where $\epsilon = \pm 1$ is the sign, $E > 0$ is the absolute value of energy, l is an integer, and

$$\psi(r, \varphi) = \begin{pmatrix} f(r) \\ g(r)e^{i\varphi} \end{pmatrix}. \tag{5}$$

Substituting (4) and (5) into (2) and taking the equalities

$$P_x \pm iP_y = -ie^{\pm i\varphi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial \varphi}\right), \tag{6}$$

into account, we obtain the following system of equations:

$$\begin{aligned} &\frac{df}{dr} - \frac{l}{r}f + \left(\epsilon E + m + \frac{Z\alpha}{r}\right)g = 0, \\ &\frac{dg}{dr} + \frac{1+l}{r}g - \left(\epsilon E - m + \frac{Z\alpha}{r}\right)f = 0. \end{aligned} \tag{7}$$

The exact solution to the Dirac equation and the discrete spectrum of energies $\epsilon E < m$ can be found in full analogy with the calculations carried out in [14]. For this purpose, we seek the functions f and g in the form [14]

$$\begin{aligned} f &= \sqrt{m+E}\, \mathrm{e}^{-\rho/2}\, \rho^{\gamma-1}(Q_1 + Q_2), \\ g &= \sqrt{m-E}\, \mathrm{e}^{-\rho/2}\, \rho^{\gamma-1}(Q_1 - Q_2), \end{aligned} \tag{8}$$

where the following notation is introduced:

$$\rho = 2\lambda r, \quad \lambda = \sqrt{m^2 - E^2}, \quad \gamma = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2}, \quad \alpha \equiv e^2. \tag{9}$$

It should be noted that the quantity γ was found by studying the behavior of wave functions at small r, and that the equality $(\gamma - 1/2)^2 - (Z\alpha E/\lambda)^2 = (l + 1/2)^2 - (Z\alpha m/\lambda)^2$ holds.

Now, transforming equation system (7) in accordance with the case of three spatial dimensions, their solution, which remains finite at $\rho = 0$, can be expressed in terms of the degenerate hypergeometric function $F(a, b; z)$ as follows:

$$\begin{aligned} Q_1 &= AF\left(\gamma - \frac{1}{2} - \left(\frac{Z\alpha E}{\lambda}\right), 2\gamma; \rho\right), \\ Q_2 &= BF\left(\gamma + \frac{1}{2} - \left(\frac{Z\alpha E}{\lambda}\right), 2\gamma; \rho\right), \end{aligned} \tag{10}$$

where the constants A and B are determined by

$$B = \frac{\gamma - 1/2 - Z\alpha E/\lambda}{l + 1/2 + Z\alpha m/\lambda} A. \tag{11}$$

The energy spectrum is determined from the equation

$$\gamma - \frac{1}{2} - \frac{Z\alpha E}{\lambda} = -n_r, \tag{12}$$

and it can be easily shown that the following values of the quantum number n_r are admissible: $0, 1, 2, \ldots$ for $l \geq 0$ and $1, 2, 3, \ldots$ for $l < 0$. Consequently, the discrete spectrum of electron energies in the field (1) has the form

$$E = m\left[1 + \frac{(Z\alpha)^2}{(n_r + \sqrt{(l+1/2)^2 - (Z\alpha)^2})^2}\right]^{-1/2}. \tag{13}$$

This energy spectrum is similar to the energy spectrum of a scalar charged particle in a Coulomb field in three spatial dimensions, because in the (2+1)-dimensional quantum theory the electron behaves like a spinless fermion. However, there is a significant difference between the spectra of a true boson and a spinless electron. For instance, the electron energy at the lower level ($l = n_r = 0$) is

$$E_0 = m\sqrt{1 - (2Z\alpha)^2}. \tag{14}$$

The energy E_0 of the lower level becomes zero for $Z\alpha = 1/2$. We recall that in three spatial dimensions the fermion energy E_0 vanishes for $Z\alpha = 1$, and the energy of a spinless charged particle at the lower level for $Z\alpha = 1/2$ is equal to $m/\sqrt{2}$. The formula for the energy of an electron ground state in the Coulomb field of a point charge becomes meaningless when $E_0(Z)$ vanishes. The corresponding wave functions oscillate with $r \to 0$, therefore no boundary condition exists at the origin [10–12].

In order to find the electron energy spectrum in the Coulomb field, we must lay down some boundary condition for $r = 0$, i.e., we should consider a potential truncated at some distance R [13], which, from the standpoint of physics, is equivalent to the accounting for the finite dimensions of the nucleus. Assuming that in the case of two spatial dimensions with growing Z over the region $2Z > 137$ the electron energy levels behave similarly to the three-dimensional case, i.e., become negative and sink to the lower continuum boundary, $-m$, we show that such a situation actually takes place.

For this purpose, let us consider the solutions to and the spectrum of the Dirac equation over the region $2Z > 137$, and determine the corresponding value of Z_{cr}. Since for finding Z_{cr} we must consider energies near the lower continuum boundary $-m$, we write the Dirac equation taking into account the fact that $\epsilon E \cong -m$. Introducing the functions $F(r) = rf(r)$ and $G(r) = rg(r)$ and eliminating $G(r)$ from system (7), we come to an equation that determines the function F near the lower continuum boundary $-m$ in the form

$$\frac{d^2F(r)}{dr^2} + \left(E^2 - m^2 + \frac{2\epsilon E Z\alpha}{r} + \frac{(Z\alpha)^2 - l(l+1)}{r^2}\right) F(r) = 0. \tag{15}$$

Note that close to the upper continuum boundary, i.e., for $\epsilon E \cong m$, after the replacement of $F(r)$ by $G(r)$ equation (15) defines the function $G(r)$.

The solution to (15) that decreases with $r \to \infty$ is expressed in terms of the Whittaker function as

$$F(r) = DW_{\beta,i\nu}(2\lambda r), \tag{16}$$

where

$$\beta = \epsilon E Z\alpha/\lambda, \quad \nu = \sqrt{(Z\alpha)^2 - (l+1/2)^2}, \quad \lambda = \sqrt{m^2 - E^2}. \tag{17}$$

The solution for $G(r)$ close to $\epsilon E = -m$ is found from the equality

$$G(r) = \frac{1}{Z\alpha}\left((1+l)F - \frac{dF}{dr}\right) \tag{18}$$

using the recursive relations for the Whittaker functions.

Near the upper continuum boundary, solution (16) with $\epsilon E = m$ describes the function $G(r)$. A bound state of the electron (with $|E| < m$ or $\lambda > 0$) is always localized in space, regardless of whether this bound state is close to the upper or to the lower continuum. This can easily be verified, using the following asymptotic form of the Whittaker functions for large values of the argument $|z|$

$$W_{\beta,\nu}(z) \sim e^{-z/2}(z)^{\beta}. \tag{19}$$

Such a behavior of the wave functions of bound electron states can be easily explained, regarding equation (15) as a one-dimensional Schrödinger equation that describes a particle with effective energy $E' = (E^2 - m^2)/2m$ in a field with the effective potential

$$U_{\text{eff}}(r) = -\epsilon E Z\alpha/mr - (Z\alpha)^2/2mr^2.$$

Note that in the case $\epsilon E = -m$ the effective potential is a wide-barrier potential (see, e. g., [12] on the behavior of the effective potential in the (3+1)-dimensional case). It is worthwhile to emphasize that in 2+1 dimensions the effective potential does not contain the term $-s(s+1) \equiv -3/4$ that depends on the electron spin. We also note that the wave functions oscillate for $|E| > m$.

The value of Z_{cr} can be found using the exact solutions obtained. Let us do it for a simple model specifying the potential as

$$\begin{aligned} A_0^Z(r) &= -Ze_0/r, \quad r \geq R; \\ A_0^Z(r) &= -Ze_0/R, \quad r \leq R. \end{aligned} \tag{20}$$

In the interval $r \leq R$ the equation for $F(r)$ has the form

$$\frac{d^2F}{dr^2} - \frac{dF}{rdr} + \left(\left(\epsilon E + \frac{Z\alpha}{R}\right)^2 - m^2 + \frac{1-l^2}{r^2}\right)F(r) = 0, \tag{21}$$

whose solution is expressed in terms of the Bessel functions ($J_n(z)$) and the Neumann functions ($N_n(z)$) of the integer index n as follows [15]:

$$F(r) = r(A_1 J_{|l|}(\varkappa r) + B_1 N_{|l|}(\varkappa r)), \tag{22}$$

where

$$\varkappa = \sqrt{\left(\epsilon E + \frac{Z\alpha}{R}\right)^2 - m^2}. \tag{23}$$

In order that $F(r)$ be finite at $r = 0$, it is necessary that $B_1 = 0$.

In order to find the Dirac equation spectrum, the solutions must be sewn at the point $r = R$

$$\left(\frac{G(r)}{F(r)}\right)_{r=R-0} = \left(\frac{G(r)}{F(r)}\right)_{r=R+0}. \tag{24}$$

In particular, for the ground state $l = 0$ with $\epsilon E = -m$, with account of the fact that the parameter R is small relative to the electron Compton length $1/m$, so that we may set $\varkappa \cong Z\alpha/R$, we obtain a transcendental equation that determines (for fixed R) the critical charge [16]

$$\frac{J_1(X)}{J_0(X)} = \left(1 - \frac{xW'_{\beta,i\nu/2}(x)}{W_{\beta,i\nu/2}(x)}\right), \tag{25}$$

where $X = Z_{\text{cr}}\alpha$, $\nu = \sqrt{(2X)^2 - 1}$, $\beta = -mZ\alpha/\lambda$, $x = \lambda R$, and the prime denotes the derivative with respect to the argument x of the Whittaker function.

The numerical solution of (25) for $Rm = 0.03$ and 0.02 respectively yields $Z_{\text{cr}} \cong 89$ and 84 [16], which is much less than for a similar model in 3+1 dimensions, in which, for instance, the critical charge for $Rm = 0.03$ is $Z_{\text{cr}} \cong 170$ [10–12]. Note also that with decreasing R the value of Z_{cr} also decreases.

Consequently, a vacuum of (2+1)-dimensional QED in a strong Coulomb field must manifest instability with respect to the electron-positron pair production at much lower value of the critical charge than in (3+1)-dimensional QED. The properties of the (2+1)-dimensional QED vacuum and the relation between the model and the Chern–Simons theory will be discussed in a forthcoming article.

REFERENCES

1. R.E. Prange and S.M. Girvin, Eds., *The Quantum Hall Effect,* 2nd edition, N.Y., 1990.
2. F. Wilczek, *Fractional Statistics and Anion Superconductivity,* Singapore, 1990.
3. A. Neagu and A.M.J. Schakel, *Phys. Rev.,* vol. D48, p. 1785, 1993.
4. A.M.J. Schakel, *Phys. Rev.,* vol. D43, p. 1428, 1991.
5. A.M.J. Schakel and C.W. Semenoff, *Phys. Rev. Lett.,* vol. 66, p. 2653, 1991.
6. S.S. Gershtein and Ya.B. Zel'dovich, *Zh. Eksp. Teor. Fiz.,* vol. 57, p. 654, 1969.
7. J. Reinhardt and W. Greiner, *Rep. Progr. Phys.,* vol. 40, p. 219, 1977.
8. J. Rafelski, L. Fulcher, and A. Klein, *Phys. Reports,* vol. C38, p. 227, 1978.
9. M. Soffel, B. Müller, and W. Greiner, *Phys. Reports,* vol. C85, p. 51, 1982.
10. Ya.B. Zel'dovich and V.S. Popov, *Usp. Fiz. Nauk,* vol. 105, p. 403, 1971.
11. A.B. Migdal, *Fermions and Bosons in Strong Fields* (in Russian), Moscow, 1978.
12. A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (in Russian), Moscow, 1988.
13. I. Pomeranchuk and Ya. Smorodinsky, *J. Phys. USSR,* vol. 9, p. 97, 1945.
14. V.B. Berestetskii, E.M. Lifshits, and L.P. Pitaevskii, *Quantum Electrodynamics* (in Russian), Moscow, 1980.
15. I.S. Gradshtein and I.M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* (in Russian), Moscow, 1963.
16. V.R. Khalilov and C.L. Ho, *Mod. Phys. Lett.,* vol. A13, p. 615, 1998.

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