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GLOBAL SYNCHRONIZATION IN A CHAIN OF PARAMETRICALLY COUPLED QUADRATIC MAPPINGS

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A chain of parametrically linearly coupled quadratic mappings was analytically shown to have global synchronization, i.e., that the system tends to a trivial equilibrium state.

INTRODUCTION

The development of the qualitative theory of differential equations shows that the behavior of many physical, chemical, and some other *distributed* systems can effectively be simulated by a network (or lattice) of mappings, i.e., by a population of interacting subsystems. Such subsystems can be both deterministic, when the next state $\mathbf{x}_{n+1}$ is uniquely determined by a sequence of preceding states $\mathbf{x}_n, \mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \ldots$, and stochastic, for which the probability of subsystem transition into a new state is specified. The dynamic systems most commonly chosen as lattice elements are the one-dimensional mappings $T_a: \ x \mapsto f(a, x)$, or, in terms of iterations,

$$x_{n+1} = f(x_n, a).$$

The coupling between the mappings can be of different types. Used most frequently is the diffusion coupling (see, e.g., [1–6]), in which each network element (i, j) interacts with its neighbors according to a diffusion law. In a two-dimensional case this coupling can be represented as

$$x_{n+1}^{i,j} = f(x_n^{i,j}, a) + d_1\Big[f(x_n^{i-1,j}, a) - 2f(x_n^{i,j}, a) + f(x_n^{i+1,j}, a)\Big] + d_2\Big[f(x_n^{i,j-1}, a) - 2f(x_n^{i,j}, a) + f(x_n^{i,j+1}, a)\Big],$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, d_1 and d_2 are the coefficients of diffusion along the horizontal and vertical directions in the lattice. In order not to introduce boundary conditions, the toroidal lattices are often considered, i.e., the superscripts i and j are defined modulo N. A typical feature of diffusion coupling is that the state of each element is as if "smoothed up" under the influence of the surrounding neighbors, and this smoothing up is the stronger, the greater the coefficients d_1 and d_2.

Another method of introducing coupling is the parametric one (see, e.g., [7]). The dynamics of the (i, j)th lattice element can then be generally represented as

$$x_{n+1}^{i,j} = f(x_n^{i,j}, a_n^{i,j}),$$

where

$$a_n^{i,j} = \varphi(a, x_n^{i,j}, x_n^{i+1,j}, x_n^{i-1,j}, x_n^{i,j-1}, x_n^{i,j+1})$$

and φ is a certain function. Parametric coupling is notable for the fact that the value of the control parameter a depends on the states of the elements adjacent of the isolated one. The dynamics of such a lattice can be interpreted as the interaction between the isolated element and a certain environment, which acts on the element parametrically. This kind of interaction is typical, in particular, of some models that describe certain chemical and biological systems (see [8, 9]).

The approximation of the original medium by a network of coupled mappings allows one to develop a rather efficient approach to the well-known problem of self-organization. In the given context, this problem can be formulated as follows: why is a complex space-time state more preferable for some nonlinear media than a simple uniform behavior (when a system passes, spontaneously as it were, from a practically homogeneous state into a spatially nonuniform one), and how can such a state be realized? In our earlier studies (see, e. g., [10–12]) based on *numerical* analysis, we proposed various mechanisms, which led to the origination of sufficiently complex spatial structures. We also studied the feasibility of the appearance of certain space-time "clusters" in the medium, consisting of identically functioning coupled mappings.

In the present article, we carry out a completely *analytical* investigation of a one-dimensional network (chain) of parametrically coupled mappings, each being capable of manifesting both regular and chaotic dynamics.

DYNAMICS OF A CHAIN OF PARAMETRICALLY COUPLED QUADRATIC MAPPINGS

Let us consider as an element, whose set represents a one-dimensional network, a quadratic mapping

$$x \longmapsto ax(1-x), \tag{1}$$

where $a \in [0, 4]$. It is well known (see, e. g., [13, 14]) that in this range of parameters such a mapping may behave both periodically and chaotically, with full topological and metric mixing. We consider a one-dimensional network (i. e., a chain) of mappings (1) with parametric coupling

$$\begin{aligned} x_1 &\longmapsto f_1(x_1, \dots, x_N)x_1(1-x_1), \\ x_2 &\longmapsto f_2(x_1, \dots, x_N)x_2(1-x_2), \\ &\dots\dots\dots\dots\dots\dots\dots\dots, \\ x_N &\longmapsto f_N(x_1, \dots, x_N)x_N(1-x_N). \end{aligned} \tag{2}$$

In order to perform analytic treatment, we demand that the values of all functions f_i lie within the interval $[0, 1]$. Let us also suppose that $f_{1+k}(x_1, x_2, \dots, x_N) \equiv f_1(x_{1+k}, x_{2+k}, \dots, x_{N+k})$, where the subscripts are taken $\bmod N+1$. The simplest case of this type of coupling is a linear dependence of each element (1) on all other elements, i. e., $f_1 = \sum_i a_i x_i$. We study separately several cases of such a coupling.

1. Coupling of the Form $f_i = ax_{i+1}$

In this case, the dynamics of the whole chain can be written as

$$\begin{aligned} x_1 &\longmapsto ax_2x_1(1-x_1), \\ x_2 &\longmapsto ax_3x_2(1-x_2), \\ &\dots\dots\dots\dots\dots, \\ x_N &\longmapsto ax_1x_N(1-x_N). \end{aligned} \tag{3}$$

It can easily be seen that for $a \in [0, 4]$ and for the initial conditions $(x_1^0, \dots, x_N^0) \in [0, 1]^N$, the phase point of chain (3) will always be inside the N-dimensional cube $[0, 1]^N$.

Let us first consider a chain consisting of only two elements, $N = 2$

$$F\colon \begin{cases} x_1 \longmapsto f(x_1, x_2) = x_2ax_1(1-x_1), \\ x_2 \longmapsto g(x_1, x_2) = x_1ax_2(1-x_2), \end{cases} \tag{4}$$

where $a \in [0,4]$. Suppose first that the state of chain (4) is synchronized, i.e., $x_1 = x_2$. It is easy to see that this equality defines an invariant manifold for mapping (4). Therefore, if $x_1 = x_2$, this mapping is one-dimensional

$$x \longmapsto ax^2(1-x). \tag{5}$$

For $0 < a < 4$, this mapping has a single fixed point $x^{(1)} = 0$. And if $a = 4$, a second fixed point $x^{(2)} = 1/2$ appears as a result of tangent bifurcation. The point $x^{(1)}$ is stable in the entire range $[0,4]$, while $x^{(2)}$ is semistable.

Let us now consider the general case of mapping (4). This mapping has no fixed points, except for those already investigated: $O_1 \equiv \left(x_1^{(1)}, x_2^{(1)}\right) = (0,0)$ and $O_2 \equiv \left(x_1^{(2)}, x_2^{(2)}\right) = (1/2, 1/2)$. Let us find the range of parametric values in which these points are stable. Because $\mathrm{D}F(O_1) = 0$, the stability of point O_1 cannot be determined by first-approximation analysis. Therefore we expand the functions f and g around this point

$$\Delta f = f_{x_1}(O_1)\Delta x_1 + f_{x_2}(O_1)\Delta x_2 + \frac{1}{2}\Big(f_{x_1x_2}(O_1) + 2f_{x_1x_2}(O_1) + f_{x_2x_2}(O_1)\Big),$$

$$\Delta g = g_{x_1}(O_1)\Delta x_1 + g_{x_2}(O_1)\Delta x_2 + \frac{1}{2}\Big(g_{x_1x_2}(O_1) + 2g_{x_1x_2}(O_1) + g_{x_2x_2}(O_1)\Big),$$

where $\Delta x_1 = x_1 - 0$, $\Delta x_2 = x_2 - 0$. Thus,

$$\Delta f = a\Delta x_1 \Delta x_2, \quad \Delta g = a\Delta x_1 \Delta x_2.$$

In the vicinity of O_1 mapping (4) becomes

$$x_1 \longmapsto a x_1 x_2, \quad x_2 \longmapsto a x_1 x_2. \tag{7}$$

Thus, in order to analyze the stability of the point O_1 under mapping (4), it suffices to explore the effect of mapping (7) in a small vicinity of this point, i.e., to consider the quantity

$$R = \rho_1 - \rho_2 = x_1^2 + x_2^2 - ({x'_1}^2 + {x'_2}^2) = x_1^2 + x_2^2 - 2a x_1^2 x_2^2,$$

where ρ_1 and ρ_2 are the distances from the point O_1 before and after one iteration, respectively. From the last relationship, we find that if $x_1^2 < 1/a^2$, $x_2^2 < 1/a^2$, then $R > 0$. This means that if these inequalities hold, then local compression takes place: the point O_1 is locally stable for all $0 < a \le 4$.

Let us now explore the behavior of mapping (4) near the point $O_2 = (1/2, 1/2)$ for $a = 4$. Then (4) becomes

$$x_1 \longmapsto 4x_2x_1(1-x_1), \quad x_2 \longmapsto 4x_1x_2(1-x_2). \tag{4$'$}$$

Expanding the right-hand side in the vicinity of O_2 up to quadratic terms, we obtain

$$\Delta x_1 \longmapsto \Delta x_2 - 2\Delta x_1^2, \quad \Delta x_2 \longmapsto \Delta x_1 - 2\Delta x_2^2, \tag{8}$$

where $\Delta x_1 = x_1 - 1/2$, $\Delta x_2 = x_2 - 1/2$. In order to analyze mapping (8) produced, let us pass to more convenient variables

$$y_1 = \Delta x_2 + \Delta x_1, \quad y_2 = \Delta x_2 - \Delta x_1.$$

One can easily see that y_1 is proportional to the deviation from the point O_2 along the diagonal $x_1 = x_2$ and y_2 is proportional to the deviation in the orthogonal direction, i.e., in the direction perpendicular to the diagonal. In the new variables the point $O_2 = \left(x_1^{(2)}, x_2^{(2)}\right) = (1/2, 1/2)$ will have the coordinates $O_2 = (y_1, y_2) = (0,0)$, and mapping (8) will become

$$G: \begin{cases} y_1 \longmapsto y_1 - (y_1^2 + y_2^2), \\ y_2 \longmapsto -y_2(2y_1 + 1). \end{cases} \tag{8$'$}$$

In order to evaluate the behavior of this mapping near O_2, let us consider its second iteration, G^2, retaining only terms of second and lower orders

$$G^2 : \begin{cases} y_1 \longmapsto y_1 - (y_1^2 + y_2^2), \\ y_2 \longmapsto y_2(1 + 4y_1). \end{cases} \tag{8''}$$

This relationship shows that both for $y_2 > 0$ and for $y_2 < 0$ the image point of mapping (8″) moves away from the point O_2 after each iteration. O_2 is an attracting point only for the invariant diagonal $x_1 = x_2$ if $\left(x_1^{(2)}, x_2^{(2)}\right) > (1/2, 1/2)$.

Thus, almost all points of the square $[0,1] \times [0,1]$ are attracted by the fixed point O_1. Consequently, the two parametrically coupled mappings display *global* synchronization, i. e., almost all trajectories tend to the trivial equilibrium state $x_1 = x_2 = 0$.

Let us now suppose that the chain has an arbitrary length $N > 2$. The investigation of synchronization $x_1 = x_2 = \ldots = x_N$ is in this case fully identical to the picture for $N = 2$. Furthermore, it is easy to show that mapping (3) has no fixed points other than $O_1 = (0, 0, \ldots, 0)$ at $0 < a \leq 4$ and $O_2 = (1/2, 1/2, \ldots, 1/2)$ at $a = 4$. The analysis of these points stability, which was carried out in exactly the same way as the above one, shows that, just as in the two-dimensional case $N = 2$, almost all iterations of the mapping converge to the point O_1.

2. Other Types of Linear Coupling

We manage to treat analytically a more complex type of parametric coupling. Thus each chain element interacts with *two* nearest neighbors. In this case, the investigation falls into two variants. The first variant does not consider the value of the chain element

$$\begin{aligned} &x_1 \longmapsto (x_N + x_2)ax_1(1 - x_1), \\ &x_2 \longmapsto (x_1 + x_3)ax_2(1 - x_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &x_N \longmapsto (x_{N-1} + x_1)ax_N(1 - x_N). \end{aligned} \tag{3'}$$

The second variant considers this value

$$\begin{aligned} &x_1 \longmapsto (x_N + x_1 + x_2)ax_1(1 - x_1), \\ &x_2 \longmapsto (x_1 + x_2 + x_3)ax_2(1 - x_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &x_N \longmapsto (x_{N-1} + x_N + x_1)ax_N(1 - x_N). \end{aligned} \tag{3''}$$

Analytical investigation of these variants shows that they do not basically differ from the case discussed above. Systems (3′) and (3″) also have only two fixed points: $O_1 = (x_1, x_2, \ldots, x_N) = (0, 0, \ldots, 0)$ and $O_2 = (x_1, x_2, \ldots, x_N) = (1/2, 1/2, \ldots, 1/2)$, and only O_2 is stable and attracts almost all trajectories from the cube $[0,1]^N$.

CONCLUSION

The deterministic chaos approach to the solution of an old problem, i. e., the description of the self-organization phenomenon, or the formation and development of complex ordered structures, has recently obtained a new impetus. Sufficiently complicated systems are known to be capable of self-organization. A necessary prerequisite of self-organization effects is the energy flow delivered to the system from an external source and dissipated by it. Owing to this flow, the system can create structures autonomously. It is obvious that self-organization effects cannot be an exclusive property of complex objects; they should also be observed in more simple systems.

Of much interest are the distributed media composed of discrete elements, which interact locally with each other. Such media describe approximately the natural spatially extended systems. It appears that even when individual system elements have complex structures, their internal complexity is not manifested in their mutual interaction and, from the standpoint of the macrosystem, they function as simple enough objects with a small number of effective degrees of freedom. Therefore the description of coupled subsystems with the aid of networks is a fully justified approach. In the present study we have shown on a sufficiently rigorous level that a chain composed of parametrically interacting one-dimensional mappings capable of manifesting both regular and chaotic dynamics, is self-synchronizing. This means that it passes into a state in which all elements are functioning identically. Consequently, if an original medium can be approximated by such a one-dimensional network of coupled nonlinear subsystems, then such a medium will evolve toward the state of full synchronism.

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