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GEOPHYSICS

CALCULATION OF ASSOCIATED MASS TENSOR IN THE PROBLEM OF EARTH'S HARD CORE MOTION

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We have calculated the associated mass tensor for harmonic oscillations of the Earth's hard core. The consideration of this tensor causes free oscillation periods of the Earth's inner core to increase by approximately an hour. For some values of the core density, the periods of oscillation are close to the experimental ones obtained by D. E. Smylie. The splitting of experimental frequencies can well be interpreted as the splitting of the equatorial mode of free oscillations of the inner core in the gravity field of the asymmetric shell, provided that the latter possesses a corresponding quadrupole moment.

1. PROBLEM FORMULATION

In [1], we have studied free oscillations of the inner core in an arbitrary direction. The problem has been solved in the Busse formulation [2] with account being additionally taken of the nonequilibrium part of the Earth's gravity field. However, to finally solve the problem it was necessary to know the magnitude of the associated mass tensor, which in numerical estimates [1] was taken to be zero. To allow for the associated mass tensor is necessary because some of the kinetic energy of the hard inner core is transmitted to the surrounding liquid, and so appropriate hydrodynamic equations are to be solved to calculate it. For polar oscillations, this problem was solved by Busse [2] under the assumption that the motion of the liquid was symmetrical about the rotation axis. But no calculations were made for oscillations in an arbitrary direction. It is the solution of the latter problem the present work is devoted to.

According to [2], the velocity field in a liquid core is the solution of the following system of equations, the liquid being assumed to be ideal,

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{V} = 2\Omega\frac{\partial \mathbf{V}}{\partial z}, \qquad \operatorname{div}\mathbf{V} = 0, \tag{1}$$

where $\mathbf{V}$ is the velocity vector and Ω is the angular velocity. The boundary conditions for an ideal liquid are that the normal velocity component on the surface bounding the liquid must equal the normal velocity component of the surface itself [3]. As applied to our case, this means that

$$(\mathbf{nV}) = 0 \quad \text{at} \quad r = r_E, \qquad (\mathbf{nV}) = (\mathbf{nV}_I) \quad \text{at} \quad r = r_I, \tag{2}$$

where $\mathbf{n}$ is the vector of the normal to the core surface, r_E is the radius of the external core boundary, r_I is the radius of the hard inner core boundary, and (nV_I) is the normal velocity component of a point on the hard core surface directed in the sense of $\mathbf{n}$. According to the Helmholtz expansion theorem [4], we seek the solution of system (1) subject to boundary conditions (2) in the form

$$\mathbf{V} = -\nabla\varphi + \operatorname{curl}\mathbf{A}, \tag{3}$$

where φ and $\mathbf{A}$ are the scalar and vector velocity potentials, respectively. It has been demonstrated in [1, 2] that the free oscillation periods of the Earth's core are small compared to the Earth's period of rotation about its axis. Therefore, the solution of the above equations can be sought for by the method of successive approximations. In doing so, we would say that the approximate solution is of order k if account was taken, when obtaining it, of all the terms in Ω up to and including order k.

2. ZEROTH-ORDER APPROXIMATION

In the zeroth-order approximation, system (1) subject to boundary conditions (2) assumes the form

$$\Delta\varphi^{(0)} = 0, \qquad (\mathbf{n}\nabla\varphi^{(0)})_{r=r_E} = 0, \quad -(\mathbf{n}\nabla\varphi^{(0)})_{r=r_I} = (\mathbf{n}\mathbf{V}_I). \tag{4}$$

This problem was solved by Lamb [5]. We present Lamb's problem solution using the following notation (according to [6]): the lower-case Latin subscripts i, j, k vary from 1 to 3; the sum is taken from 1 to 3 with respect to terms in which an index appears twice (Einstein's summation convention); ε_{ijk} is the Levi–Civita symbol [7] antisymmetric in any pair of indices and with $\varepsilon_{123} = +1$; the symmetrical and trace-free (STF) part of the tensor M_{ijk} is designated $M_{<ijk>}$ [8]; $f_{,i} = \partial f/\partial x^i$; $r^2 = x^2 + y^2 + z^2 = x_i x_i$; $n_i = x_i/r$.

With this symmetrical notation, expression (4) assumes the following form: $\varphi^{(0)}_{,ii} = 0$, $n_i\varphi^{(0)}_{,i} = 0$ at $r = r_E$; $-n_k\varphi^{(0)}_{,k} = V_{Ij}n_j$ at $r = r_I$.

The boundary conditions dictate the solution in the form

$$\varphi^{(0)} = \left(\frac{M_i^{(0)}}{r^2} + r m_i^{(0)}\right) n_i, \quad V_i^{(0)} = -m_i^{(0)} - \frac{M_i^{(0)} - 3M_k^{(0)} n_k n_i}{r^3}, \tag{5}$$

where $m_i^{(0)}$ and $M_i^{(0)}$ are time-varying functions defined from the boundary conditions. Substituting (5) into (2) and collecting the like terms, we get

$$M_k^{(0)} = \frac{V_{Ik} r_E^3 r_I^3}{2(r_E^3 - r_I^3)}, \quad m_k^{(0)} = \frac{V_{Ik} r_I^3}{r_E^3 - r_I^3}.$$

According to [2], the mass density α_{ik} of the associated mass tensor is defined to satisfy the condition

$$\frac{dT_E}{dt} = \alpha_{jk} V_{Ij} \frac{dV_{Ik}}{dt} m_I, \tag{6}$$

where

$$T_E = \sigma_E \int \frac{V^2}{2} d\tau \tag{7}$$

is the kinetic energy of the liquid, m_I is the mass of the inner core of the Earth, σ_E is the density of the homogeneous liquid core, $d\tau$ is the volume element, and integration is extended over the entire volume occupied by the liquid.

Substituting (5) into (7) with due regard for (6) in the zeroth-order approximation, we obtain

$$\alpha_{kp}^{(0)} = \frac{\sigma_E}{\sigma_I} \frac{2r_I^3 + r_E^3}{2(r_E^3 - r_I^3)} \delta_{kp}, \tag{8}$$

where δ_{kp} is the Kronecker symbol.

Formula (8) coincides with Lamb's result, which is the zeroth-order approximation for our case and exists in the nonrotating liquid.

3. FIRST APPROXIMATION

We seek the solution in the form

$$V_i = V_i^{(0)} + V_i^{(1)} = V_i^{(0)} - \nabla_i \phi^{(1)} + \varepsilon_{ijk} \nabla_j A_k^{(1)}, \tag{9}$$

where $\text{curl}_i\, A = \varepsilon_{ijk} \nabla_j A_k$. We introduce the notation $\widetilde{\phi} \equiv \widetilde{M_k} n_k / r^2$, where $\widetilde{M_k} = \int M_k^{(0)}(t)\, dt$. Such an integral exists if V_{Ik} is the periodic time-varying function. Substituting (9) into the first equation of system (1), we arrive at the equation

$$\frac{\partial}{\partial x_i} \left(\frac{\partial A_k^{(1)}}{\partial x_k} + 2\Omega \frac{\partial \widetilde{\phi}}{\partial z} \right) = \Delta A_i^{(1)}. \tag{10}$$

According to (3), the vector potential $A_i^{(1)}$ is determined accurate to within the gradient of some calibration function ψ. We select this function such that

$$\Delta A_i^{(1)} = 0. \tag{11}$$

This selection is possible because the left-hand side of equation (10) is the gradient of some function. Then, considering the form of $\widetilde{\phi}$, we have

$$A_k^{(1)} = \left(0,\ 0,\ \frac{-2\Omega \widetilde{M}_p n_p}{r^2} \right). \tag{12}$$

According to the form of boundary conditions (2), we seek the potential $\phi^{(1)}$ in the form $\phi^{(1)} = r m_k^{(1)} n_k + M_k^{(1)} n_k / r^2$. We substitute this expression along with (12) into (9) and then substitute the expression obtained into boundary conditions (2). Collecting the like terms, we obtain the following result

$$m_k^{(1)} = 0, \quad M_k^{(1)} = \Omega \varepsilon_{ijz} \widetilde{M_j}. \tag{13}$$

Substituting (13) and (12) into (9) and grouping the like terms, we get the following formula for velocity

$$V_i^{(1)} = -\frac{3\Omega}{r^3} \left(\varepsilon_{ijz} \widetilde{M_j} - \varepsilon_{pjz} \widetilde{M_j} n_p n_i - 2\varepsilon_{ijz} n_j \widetilde{M_k} n_k \right). \tag{14}$$

Having substituted (9), with due regard for (14), into (7), we find that the kinetic energy component in first order in Ω is

$$T^{(1)} = \sigma_E \int V_k^{(0)} V_k^{(1)}\, d\tau = 0.$$

Consequently, based on (6),

$$\alpha_{kp}^{(1)} = 0. \tag{15}$$

4. SECOND APPROXIMATION

In the second approximation, we find the velocity in the form

$$V_i = V_i^{(0)} + V_i^{(1)} + V_i^{(2)}. \tag{16}$$

Substituting (16) into (1), we obtain

$$\frac{\partial}{\partial t} \varepsilon_{ijk} \nabla_j V_k^{(2)} = 2\Omega \frac{\partial}{\partial z} (\varepsilon_{ijk} \nabla_j A_k^{(1)} - \nabla_i \phi^{(1)}), \quad \Delta V_k^{(2)} = 0, \tag{17}$$

whence it follows that the solution may be sought for in the form

$$V_i^{(2)} = 2\Omega \frac{\partial \widetilde{A}_i^{(1)}}{\partial z} + \widetilde{V}_i^{(2)},$$

where $\widetilde{V}_i^{(2)} = -\nabla_i \phi^{(2)} + \varepsilon_{ijk}\nabla_j A_k^{(2)}$, $\widetilde{A}_i^{(1)} = \int A_i^{(1)}(t)\,dt$.

In this case, the problem for $A_i^{(2)}$ coincides with (10) if $A_i^{(1)}$ is replaced by $A_i^{(2)}$ and $\phi^{(0)}$ is replaced by $\phi^{(1)}$, where $\widetilde{\phi}^{(1)} = \widetilde{M}_k^{(1)} n_k / r^2$ and $\widetilde{M}_k^{(1)} = \int M_k^{(1)}(t)\,dt$. As problem (10) is already solved, we write the answer without any intermediate explanations

$$A_k^{(2)} = \left(0,\ 0,\ \frac{-2\Omega \widetilde{M}_p^{(1)} n_p}{r^2}\right) = \left(0,\ 0,\ \frac{-2\Omega^2 \varepsilon_{ijz} \widetilde{\widetilde{M}}_j n_i}{r^2}\right), \tag{18}$$

where

$$\widetilde{\widetilde{M}}_k = \iint \dot{M}_k^{(0)}(t)\,dt^2. \tag{19}$$

The form of the harmonic function was obtained from boundary conditions (2) by substituting thereinto expression (16), consideration being given for (18)

$$\phi^{(2)} = \frac{M_k^{(2)} n_k}{r^2} + \frac{J_{\langle ijk\rangle}^{(2)} n_i n_j n_k}{r^4} + r^3 m_{\langle ijk\rangle}^{(2)} n_i n_j n_k, \tag{20}$$

where $M_i^{(2)} = -(\Omega^2/5)(7\widetilde{\widetilde{M}}_z \delta_{iz} + \widetilde{\widetilde{M}}_i)$, $J_{\langle ijk\rangle}^{(2)} = 3\Omega^2 \widetilde{\widetilde{M}}_{\langle i}\delta_j^z \delta_{k\rangle}^z r_E^2 r_I^2 (3r_E^5 - r_I^5)/(3r_E^7 - r_I^7)$, $m_{\langle ijk\rangle}^{(2)} = 12\Omega^2 \widetilde{\widetilde{M}}_{\langle i}\delta_j^z \delta_{k\rangle}^z (r_E^2 - r_I^2)/(3r_E^7 - r_I^7)$.

The expression for the kinetic energy component in order 2 in Ω was obtained by substituting expression (16) into (7)

$$T^{(2)} = \sigma_E \int \left(\frac{{V^{(1)}}^2}{2} + V_k^{(0)} V_k^{(2)}\right) d\tau. \tag{21}$$

Using the orthogonality condition of spherical functions differing in order and substituting (16) into (21) with due regard for equations (18) through (20) and (14), we get the following expressions:

$$\begin{aligned} \alpha_{11}^{(2)} &= \alpha_{22}^{(2)} = \frac{29}{10}\frac{\Omega^2}{\chi^2}\frac{r_E^3}{r_E^3 - r_I^3}\frac{\sigma_E}{\sigma_I}, \\ \alpha_{33}^{(2)} &= -\frac{12}{10}\frac{\Omega^2}{\omega_p^2}\frac{r_E^3}{r_E^3 - r_I^3}\frac{\sigma_E}{\sigma_I}, \\ \alpha_{ij}^{(2)} &= 0 \quad \text{at} \quad i \neq j. \end{aligned} \tag{22}$$

In calculating expressions (22), account was also taken of the fact that the coordinates of the center of the Earth's hard core vary, according to [1], as

$$x = A\cos\chi t, \quad y = A\sin\chi t, \quad z = B\cos\omega_p t.$$

5. NUMERICAL RESULTS

The formulas were obtained in [1] for the calculation of frequencies of natural oscillations of the Earth's core in an arbitrary direction

$$\begin{aligned} \omega_j^{(k)} &= \chi_k + (-1)^j \frac{(6A_{22} + (-1)^j A_{20})\Sigma}{2\chi_k r_E^3 A_e}, \\ \omega_3 &= \omega_p - \frac{A_{20}\Sigma}{\omega_p r_E^3 A_p}, \end{aligned} \tag{23}$$

where A_{22} and A_{20} are the coefficients of expansion of the anomalous part of the gravity potential into a series in spherical functions [1], the subscript k assumes the values $\pm$, $\chi_\pm = -\Omega' \pm \sqrt{\Omega'^2 + \omega_e^2}$, $\Omega' = \Omega/A_e$,

$\omega_p^2 = (4/3A_p)\pi G\sigma_E\Sigma$, $\omega_e^2 = (4/3A_e)(\pi G\sigma_E - \Omega^2)\Sigma$, G is the gravitational constant, $\Sigma = 1 - \sigma_E/\sigma_I$, $A_{e,p} = 1 + (\sigma_E/\sigma_I)\alpha_{e,p}$, $j = 1, 2$, $\alpha_e = \alpha_{11} = \alpha_{22}$ and $\alpha_p = \alpha_{33}$.

The effect of the associated mass tensor manifests itself in that the free oscillation periods of the Earth's hard core grow longer (Table 1). The periods in the table are designated in accordance with [1]. The first line lists the periods calculated according to [1] with no consideration for the associated mass tensor, and the second, with due regard for the associated mass tensor for densities of $\sigma_E = 12\,000$ kg/m^3 and $\sigma_I = 12\,597$ kg/m^3. The equatorial mode got split into 4 harmonics ($T_1^{(+)}$, $T_2^{(+)}$, $T_1^{(-)}$ and $T_2^{(-)}$), the splitting of the oscillation period into two components offset by approximately ± 0.5 h from the initial one is caused by the rotation of the coordinate system with a period of 24 h and that into the other two components offset by approximately ± 0.001 h, by the gravity field due to the asymmetric shell of the Earth.

Table 1

Free Oscillation Periods (in Hours) of the Inner Core of the Earth for Some Values of Hard Core Density σ_I at $\sigma_E = 12\,000$ kg/m^3, and for the Associated Mass Tensor

σ_I, kg/m^3	$T_1^{(+)}$	$T_1^{(-)}$	$T_2^{(+)}$	$T_2^{(-)}$	T_3
	5.253266	3.653644	5.251942	3.653197	4.377025
12 597	6.382047	4.715489	6.380558	4.714872	5.317058
12 960	4.848023	3.808195	4.846950	3.807668	4.212587
13 702	4.589068	3.643048	4.588062	3.642539	4.015690

The frequencies calculated by formulas (23) depend on the average densities of the hard core and of the liquid core. The oscillation periods for the densities $\sigma_E = 12\,000$ kg/m^3 and $\sigma_I = 12\,960$ kg/m^3 are listed in the third line of Table 1, and those for the densities $\sigma_E = 12\,000$ kg/m^3 and $\sigma_I = 13\,072$ kg/m^3, in the fourth line. In this case, the equatorial period T_3 coincides, within the accuracy of measurement, with that obtained experimentally in [9] $T = 4.015 \pm 0.001$ h, and the periods $T_1^{(-)}$ and $T_2^{(-)}$ prove to be close to the data obtained experimentally by means of superconducting gravimeters [9]: 3.5820 ± 0.0008 h and 3.7677 ± 0.0006 h. Though approximating each other, the average periods $(T_1^{(-)} + T_2^{(-)})/2 = 3.642793$ h and 3.6748 ± 0.0007 h fail to coincide within the accuracy of measurement. However, average frequencies can be made to agree by appropriately selecting the densities σ_E and σ_I. There being no accurate data on these parameters so far, one cannot rule out on these grounds the possibility of interpreting the periods obtained in [9] as natural oscillation periods of the Earth's core.

CONCLUSIONS

Let us sum up the results of this investigation.

1. The allowance for the associated mass tensor in calculating free oscillations of the inner core of the Earth in an arbitrary direction leads to an increase by approximately 1 h of the periods of free oscillations of the Earth's core.

2. The theoretical periods of free oscillations for some core density values are close to the periods obtained as a result of processing the gravimetric observation data [9].

3. The experimental periods [9], despite their being closely allied to those calculated in this work, cannot be explained by free oscillations of the inner core within the framework of the model used. They can be explained as a result of splitting the equatorial mode of free oscillations of the inner core in the gravity field of the asymmetric shell, provided the quadrupole moment of the latter is 40 times that taken in [1].

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