*Moscow University Physics Bulletin*
*Vol. 55, No. 2, pp. 52–55, 2000*

*Vestnik Moskovskogo Universiteta. Fizika*
*UDC 534:532.527*

# ACOUSTICS AND MOLECULAR PHYSICS

# SPECTRAL DENSITY OF VORTICAL VELOCITY PULSATIONS IN A WELL-DEVELOPED FREE TURBULENT FLOW

S. G. Mikhailov

**Based on the assumption put forward as to the local character of vorticity in a free incompressible liquid flow, the vortex element of a turbulent flow is isolated and its properties are evaluated. Good agreement is found to exist between the form of the spectral density of the velocity pulsations arising as a result of the uniform transport of a random ensemble of quasistationary turbulence vortex elements by the mainstream flow and the well-known approximations of experimental data.**

There are various models of a free turbulent incompressible liquid flow that make it possible to estimate the statistical characteristics of velocity and pressure pulsations therein [1]. These models, however, fail to reveal the mechanism of interaction between such a flow and a solid body placed therein. For this reason, the pseudosonic pulsations received by sound vibration pickups placed in a turbulent flow were theoretically studied on the basis of a simplified notions of turbulence as a plane harmonic wave [2, 3]. In this work, we consider a vortical model of turbulent pulsations, within the framework of which the formation mechanism of pseudosonic noise interfering with sound reception can be verified.

To analyze the motions of a viscous incompressible liquid in the absence of free boundaries, use is made of the equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\,\mathrm{grad}\,p - \nu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}, \tag{1}$$

that relates the liquid velocity $\mathbf{u}$ and pressure $p$. Here $\rho$ and $\nu$ are the density and kinematic viscosity of the liquid, respectively. There are known a few types of flow for which the exact solutions of this equation have been found. Therefore, the major advances in hydrodynamics in studying flows over various bodies have been based on the proposal made by L. Prandtl that the entire region occupied by the liquid can be subdivided into a thin boundary layer wherein the effect of the velocity field vorticity ($\mathrm{curl}\,\mathbf{u}$) is essential and an external region wherein the liquid motion is taken to be potential [4].

Based on the Prandtl hypothesis, we assume that in a free turbulent motion, too, vorticity is other than zero only in a small part of space represented by vortex tubes. It has been known from Helmholtz's second theorem [4] that vortex tubes cannot terminate in the liquid: they either rest on the bounding surfaces or form closed rings.

Consider the velocity field formed by a single vortex ring. Let the average ring radius $R_0$ be much greater than the tube radius $r_0$. The flow inside the vortex tube is completely determined by the vorticity distribution over its cross section. In a most simple case where $\mathrm{curl}\,\mathbf{u} = \mathrm{const}$, the liquid rotates as a single whole, and the flow velocity is proportional to the distance $r'$ from the tube axis, $r' \ll r_0$. When $r_0 \ll r' \ll R_0$, the velocity field of the vortex ring approaches that of an infinite rectilinear vortex filament around which there is formed a flow known as a potential vortex. In this region, the flow velocity is inversely proportional to the distance from the vortex tube axis:

$$v(r') = \frac{\Gamma}{2\pi r'},$$

where $\Gamma$ is the velocity circulation. For $r' \gg R_0$, the potential of the ring approaches that of a dipole by virtue of the closed character of the vortex tube. A dipole region is characterized by a decrease of the flow velocity therein in proportion to the cube of the distance from the tube axis.

Let us select a function approximating the above relationships in all the three regions. We take as a basis the velocity field produced by a potential vortex. We will allow for the behavior of the velocity field at small and large distances from the axis by means of additional factors. In that case, the velocity

$$v(r') = \frac{\Gamma}{2\pi r'} \frac{1-\exp(-r'^2/r_0^2)}{1+(r'/R_0)^2},$$

is directed along the tangent to the circumference of radius $r' = \text{const}$. In the above expression, $r_0$ and $R_0$ are respectively the distances at which the effect of viscosity and the spatial structure of the vortex ring manifest themselves. As $R_0 \to \infty$, the function selected tends to the exact solution of equation (1), which is sometimes referred to as the Gaussian vortex [4, 5] wherein $r_0$ varies with time, $r_0^2 = 4\nu t$, thus pointing to a gradual reduction of its growth rate.

As a model of vortical turbulent pulsations, we consider a process formed as a result of the transport of turbulence elements randomly arranged in space by the mainstream flow having a constant velocity of $\mathbf{U}$. Each turbulence element is a motion whose vorticity is essentially other than zero only in a limited region of space, namely, in an infinite rectilinear vortex tube. The axes of the elements have different directions in space. The velocity fields produced by the turbulence elements are time-invariable, similar to one another, and differ only by the scale factor

$$u_i(r') = \frac{\Gamma_i}{\Gamma} v(r'),$$

the mathematical expectation of $\Gamma_i/\Gamma$ being equal to zero and its dispersion to unity. The pulsation velocity produced by such a process at the point with the radius-vector $\mathbf{r} = (x, y, z)$ is

$$\mathbf{u}_{\Sigma}(\mathbf{r}) = \sum_{i=-\infty}^{\infty} \mathbf{u}_i(\mathbf{r} - \mathbf{r}_i - t\mathbf{U}),$$

where $\mathbf{r}_i = (\zeta_i, \eta_i, \xi_i)$ is the position of the $i$th turbulence element at the time instant $t = 0$. Here the motion velocities of the elements, both caused by the action of the adjacent elements and self-induced, are considered negligibly small. The motion described is a model of frozen uniform and isotropic turbulent pulsations.

Let the flow velocity $\mathbf{U}$ be directed along the $x$-axis. In that case, the longitudinal component of the velocity induced at the point $(x, y, z)$ by a turbulence element having a circulation of $\Gamma$ may be written in the form

$$v_t(t) = \frac{\Gamma}{2\pi} \frac{r \sin\alpha}{x_0^2 + r^2} \frac{1 - \exp[-(x_0^2 + r^2)/r_0^2]}{1 + (x_0^2 + r^2)/R_0^2},$$

where $r = \sqrt{(y-\eta)^2 + (z-\xi)^2}$, $x_0 = (x - \xi - tU)\sin\alpha$, and $\alpha$ is the angle between the axis of the element and the $x$-axis. Its Fourier transform is

$$v_t(\omega) = \frac{\Gamma e^{(i\omega/U)(\zeta - x)}}{2U} \left\{ e^{-kr} - \frac{1}{2}\left[ e^{-kr} \operatorname{erfc}\left(\frac{r}{r_0} - \frac{kr_0}{2}\right) + e^{kr} \operatorname{erfc}\left(\frac{r}{r_0} + \frac{kr_0}{2}\right)\right] - \frac{r}{R} e^{-kR} + \frac{r}{2R} e^{R_0^2/r_0^2} \right.$$
$$\left. \times \left[ e^{-kR} \operatorname{erfc}\left(\frac{R}{r_0} - \frac{kr_0}{2}\right) + e^{kR} \operatorname{erfc}\left(\frac{R}{r_0} + \frac{kr_0}{2}\right)\right]\right\},$$

where $k = \omega/U \sin\alpha$ and $R = \sqrt{r^2 + R_0^2}$. The above assumptions allow us to use the well-known method [6] to calculate the spectral density $\Phi_{tt}(\omega)$ of velocity pulsations. As a result, we have

$$\Phi_{tt}(\omega) = \frac{2U}{V} \int_0^\infty \int_0^\pi |v_t(\alpha, r, \omega)|^2 r \, dr \, d\alpha,$$

where $V$ is the average volume of space occupied by a single turbulence element.

The plots of the normalized spectral density of velocity pulsations shown in Fig. 1 make it possible to single out three regions wherein the properties of this function are different. In the low-frequency region ($\omega \ll U/R_0$) corresponding to dipole motion, the spectral density of velocity pulsations is constant. In the

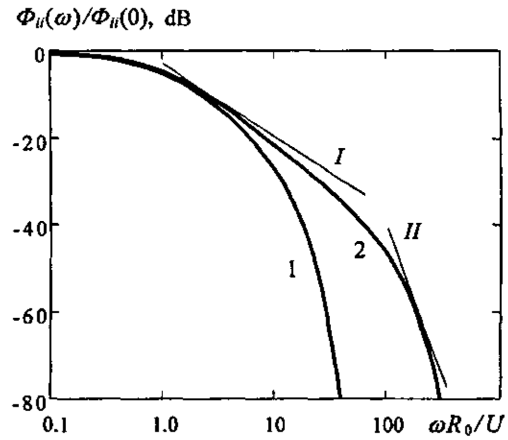


**Fig. 1**

Spectral density of vortical velocity pulsations in an isotropic turbulent flow at various external-to-internal radius ratios: (*1*) $R_0/r_0 = 10$; (*2*) $R_0/r_0 = 100$. Lines *I* and *II* are approximations by the relations $\sim \omega^{-5/3}$ and $\sim \omega^{-7}$, respectively.

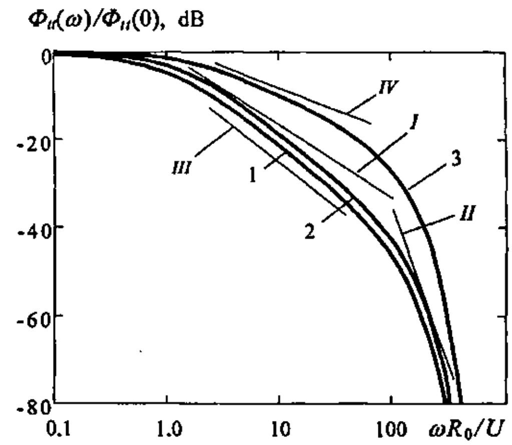


**Fig. 2**

Spectral density of longitudinal vortical velocity pulsations at $R_0/r_0 = 100$ in an isotropic turbulent flow (curve *1*) and in an anisotropic flow with turbulence element axes normal to the $x$-axis (curve *2*) and with turbulence element axes normal to the $xy$ plane (curve *3*). Lines *I*, *II*, *III*, and *IV* are approximations by the relations $\sim \omega^{-5/3}$, $\sim \omega^{-7}$, $\sim \omega^{-2}$, and $\sim \omega^{-1}$, respectively.

vortex motion frequency scale region $U/R_0 \ll \omega \ll U/r_0$ at $R_0/r_0 = 100$, the spectral density of velocity pulsations can be approximated well enough by the relation $\omega^{-5/3}$ corresponding to the inertial interval of the locally isotropic Kolmogorov–Obukhov turbulence [5]. At high frequencies ($\omega \gg U/r_0$) corresponding to motions in the viscous region, there is observed a rapid decrease of the spectral density of velocity pulsations. This decrease in the initial section can be approximated well enough by the relation $\sim \omega^{-7}$ corresponding to the dissipation interval of the locally isotropic turbulence.

We have also calculated the spectral density of vortical velocity pulsations in some types of anisotropic flows (Fig. 2). In the cases considered, the character of the frequency dependence of the spectral density remains the same as in Fig. 1, and it is only the ratio between the energies corresponding to the dipole and vortex motions (curves *1* and *2*) that changes, and the slope of the spectral density curve decreases in the vortex interval in the case of plane flow (curve *3*). It should be noted that in actual flows there are observed both velocity pulsation spectra close to those predicted by the Kolmogorov–Obukhov model (lines *I* and *II*) and ones noticeably different from them [7] (lines *III* and *IV*). The vortical model is versatile enough to describe these differences.

Since it is a random ensemble of elements and the velocity fields of each one of them that are specified (up to within a factor) in the vortical model, it seems possible to find in the future the statistical characteristics of the appropriate model of the field of vortical pressure pulsations in a free turbulent flow and to study the mechanism whereby pseudosonic pulsations form on the surface of sound vibration pickups.

## REFERENCES

1. A.S. Monin and A.M. Yaglom, *Statistical Hydromechanics* (in Russian), Moscow, pt. 1, 1992; pt. 2, 1996.
2. D.I. Blokhintsev, *Acoustics of Inhomogeneous Moving Medium* (in Russian), Moscow, 1981.
3. M. Strasberg, *J. Acoust. Soc. Am.*, vol. 66, no. 5, p. 1487, 1979.
4. L.G. Loitsyanskii, *Fluid Mechanics* (in Russian), Moscow, 1978.
5. A.S. Monin and R.V. Ozmidov, *Oceanic Turbulence* (in Russian), Leningrad, 1981.
6. B.R. Levin, *Theoretical Fundamentals of Statistical Radio Engineering* (in Russian), vol. 1, Moscow, 1974.
7. B.A. Kader and A.M. Yaglom, in: V.V. Struminskii, ed., *Turbulent Flow Problems* (in Russian), p. 65, Moscow, 1987.

14 April 1999 Department of Acoustics