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BRIEF COMMUNICATIONS

THEORETICAL AND MATHEMATICAL PHYSICS

STATISTICAL PROPERTIES OF FIXED POINTS OF DYNAMIC SYSTEMS

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A relative prevalence of fixed points of various classes in the phase plane for autonomous systems of the general form is studied. It is shown that saddle-points are most frequently met, foci are more seldom, and knots are even more seldom.

Investigating properties of the equilibrium states, i.e., fixed points, defined by relations $F(\{x_i\}) = 0$, is the first and most important stage of qualitative study of finite-dimensional dynamic systems given by a system of ordinary differential equations

$$\frac{dx_i}{dt} = F_i(\{x_i\}), \quad 1 \le i \le K. \tag{1}$$

Fixed points are classified by the magnitudes and signs of the real and imaginary parts of the characteristic indices of the system of dynamic equations (1) linearized near the equilibrium state. For systems in the phase plane ($K = 2$), the following classes of fixed points are singled out: center ($\mathrm{Re}\,\lambda_{1,2} = 0$, $\mathrm{Im}\,\lambda_{1,2} \neq 0$), focus ($\mathrm{Re}\,\lambda_{1,2} \neq 0$, $\mathrm{Im}\,\lambda_{1,2} \neq 0$), saddle-point ($\mathrm{Re}\,\lambda_{1,2} \neq 0$, $\mathrm{Im}\,\lambda_{1,2} = 0$, $\lambda_1\lambda_2 < 0$), and knot ($\mathrm{Re}\,\lambda_{1,2} \neq 0$, $\mathrm{Im}\,\lambda_{1,2} = 0$, $\lambda_1\lambda_2 > 0$) [1, pp. 19–24].

In the present note, the relative prevalence of points of these classes in the dynamic systems of a general form is studied.

The system of equations linearized near a fixed point has the form

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy. \tag{2}$$

By an appropriate choice of the time scale the following condition can be fulfilled:

$$a^2 + b^2 + c^2 + d^2 = 1. \tag{3}$$

Thus, the space of fixed points is mapped onto the surface of a 4-dimensional sphere of unit radius. It is the measures of the domains of the surface corresponding to various classes of points that determine their prevalence. An analogous method of metrization was used by Kac in estimating an average number of real roots of the polynomial of the given degree [2].

Points of the center type are not rigid (i.e., with small variation of parameters of the general form, they are transformed to points of different classes) and have a zero measure. The domain corresponding to foci is defined by the condition

$$(a - d)^2 + 4bc < 0. \tag{4}$$

The domain corresponding to saddle-points is defined by the condition

$$(a-d)^2 + 4bc > 0, \quad ad - bc < 0. \tag{5}$$

The remaining part of the hypersphere corresponds to knots.

Calculation of volumes for domains (4) and (5) on the hypersphere surface (3) was carried out by the Romberg numerical integration method for four different positions of the coordinate system axes. The obtained values of the probabilities for foci (P_F), saddle-points (P_S) and knots (P_N) are as follows:

$$P_F = 0.2920(6), \quad P_S = 0.5010(14), \quad P_N = 0.2070(20). \tag{6}$$

The figures in parentheses correspond to errors in units of the last decimal digit. Thus, for systems of the general form, the most probable type of a fixed point is the saddle-point, less probable is the focus, and the least prevailing are the knots. Frequencies of stable and unstable points for foci and knots are equal, which is evident for symmetry reasons.

The above approach can be extended to dynamic systems with a phase space of dimension $K > 2$. However, fast increase of the dimensionality of the sphere D with growth of K ($D = K^2 - 1$) and complication of the type of conditions for the domains corresponding to fixed points of various types, makes the corresponding numerical integration very difficult.

REFERENCES

1. M.I. Rabinovich and D.I. Trubetskov, *Introduction to the Theory of Oscillations and Waves* (in Russian), Moscow, 1984.
2. M. Kac, *Probability and Related Topics in Physical Sciences*, New York, 1957.

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