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GRAVITATIONAL LENSING ON A BRANE

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An expression for the angle of light deflection in the gravitational field of a global monopole embedded in the Randall–Sundrum universe has been obtained. It has been demonstrated that the deflection angle depends on the characteristic parameter k of the model, and thus, limitations for this parameter may in principle be obtained from astronomical observations.

INTRODUCTION

Recently, it has become popular to consider brane world models based on a hypothesis that our world is a hyperplane (brane) embedded in a certain higher dimensional fundamental space (see a review [1] and the references there). The number of additional dimensions and their typical size can be fairly different in various models. At the same time, as a rule, they assume that this size is large enough, and additional dimensions can in principle be detected in planned experiments in the near future.

In this paper, we consider the Randall–Sundrum universe of the second type (RS2-model) with a global monopole placed on a brane. We also study the properties of the effect of gravitational lensing that are due to the presence of an additional dimension. The choice of the subject of investigation is explained by the fact that the global monopole is one of the topological defects most interesting for cosmology. Moreover, the lensing effect is most promising for detecting topological defects in the observable part of the Universe. A similar process in the field of another defect, i. e., a cosmic string important for cosmology, was considered in [2].

GRAVITATIONAL FIELD OF A MONOPOLE ON A BRANE

In the present section, we recall the main results of [3] studying the gravitational field of a global monopole placed on a brane of the Randall–Sundrum universe with one infinite additional spatial dimension [4].

First of all, we note that since in the framework of the present model, the fields of the Standard Model forming a monopole are assumed to be localized on a brane, then, in finding the monopole metric in the linear approximation, the expression for the energy–momentum tensor should be taken in the form it has in the Minkowski space [5]. In Cartesian coordinates, its nonvanishing components have the form

$$t_{00} = \frac{\eta^2}{R^2}, \quad t_{ik} = -\eta^2 \frac{x_i x_k}{R^4}, \quad R^2 = x^i x^i, \quad i,k = 1,2,3, \tag{1}$$

where η is the energy scale of spontaneous symmetry violation that initializes the defect in question.

If there is matter on the brane, the metric of the entire five-dimensional $RS2$ space can be represented as (for further details see [6])

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu + 2N_\mu dx^\mu dy + (1+\phi)\, dy^2,$$

where

$$\widehat{g}_{\mu\nu} = e^{-2k|y|}\left(\eta_{\mu\nu} + h_{\mu\nu}\right),$$

x^μ are the coordinates on the brane ($\mu, \nu, \cdots = 0, 1, 2, 3$); the coordinate y corresponds to an additional dimension; $\widehat{g}_{\mu\nu}(x, y)$ is the metric on the time-like hyper-surface $y = \text{const}$ (the brane is located at the point $y = 0$); $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$; and k is the characteristic parameter of the theory that uniquely determines the five-dimensional cosmological constant and the brane tension entering the action [4].

It can be shown [6] that, upon applying the gauge

$$N_\mu = -\frac{\text{sgn } y}{8k} h_{,\mu}, \quad \phi = -\frac{\text{sgn } y}{4k} h_{,y}, \quad \widetilde{h}^{;\mu}_{\mu\nu} = 0,$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$ and $\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} h$ is the traceless part of $h_{\mu\nu}$, the equation for linear perturbations of the metric takes the form

$$\partial_y\left(e^{-2k|y|}\partial_y \widetilde{h}_{\mu\nu}\right) - 2k \,\text{sgn}\, y \; e^{-2k|y|}\partial_y \tilde{h}_{\mu\nu} + \Box \widetilde{h}_{\mu\nu} = -16\pi G_5 \delta(y)\left[t_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\right) t\right], \tag{2}$$

and for the trace of h, the following relation holds:

$$\Box h|_{y=0} = \frac{32 G_5 k}{3} t. \tag{3}$$

Here $\Box = \partial_\mu \partial^\mu$, G_5 is the five-dimensional gravitational constant related to the gravitational constant on the brane as $G_4 = k G_5$, and $t = \eta^{\mu\nu} t_{\mu\nu}$.

On the brane surface at $y = 0$, the solution of equations (2) and (3) can be represented as [3]

$$\widetilde{h}_{00}(\mathbf{x}) = \frac{16 G_4 \pi^3 \eta^2}{3k} \int \frac{d^3 q}{(2\pi)^3} \frac{e^{i\mathbf{qx}}}{q^2} \frac{K_2(q/k)}{K_1(q/k)},$$

$$\widetilde{h}_{ij}(\mathbf{x}) = \frac{8 G_4 \pi^3 \eta^2}{3k} \int \frac{d^3 q}{(2\pi)^3} \frac{e^{i\mathbf{qx}}}{q^2} \left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) \frac{K_2(q/k)}{K_1(q/k)},$$

the other components vanish, and

$$h(\mathbf{x}) = \frac{128 G_4 \pi^3 \eta^2}{3} \int \frac{d^3 q}{(2\pi)^3} \frac{e^{i\mathbf{qx}}}{q^3}.$$

Changing to spherical coordinates gives

$$h_{00}(R) = \frac{8\pi G_4 \eta^2}{3k} \int_0^\infty dq \frac{\sin(qR)}{qR} \frac{K_0(q/k)}{K_1(q/k)},$$

$$h_{RR}(R) = \frac{8\pi G_4 \eta^2}{3k} \int_0^\infty dq \left(\frac{2k}{q} \frac{\sin(qR)}{qR} + \frac{K_2(q/k)}{K_1(q/k)} \left(\frac{\sin(qR) - qR\cos(qR)}{q^3 R^3}\right)\right),$$

$$h_{\theta\theta}(R) = \frac{4\pi G_4 \eta^2}{3k} R^2 \int_0^\infty dq \left(\frac{4k}{q} \frac{\sin(qR)}{qR} + \frac{K_2(q/k)}{K_1(q/k)} \left(\frac{-\sin(qR) + qR\cos(qR) + q^2 R^2 \sin(qR)}{q^3 R^3}\right)\right),$$

$$h_{\varphi\varphi}(R) = h_{\theta\theta}(R) \sin^2\theta.$$

The monopole metric on a brane can be put into a more convenient form if a new radial variable r is introduced relating to R as follows:

$$h_{\theta\theta}(R) + R^2 = \left[1 - 8 G_4 \pi \eta^2 (1 + f_1(kr))\right] r^2.$$

In terms of the new radial coordinate, we have

$$ds^2 = -dt^2\left[1 - 8G_4\pi\eta^2 f_2(kr)\right] + dr^2 + \left[1 - 8G_4\pi\eta^2(1 + f_1(kr))\right] r^2 d\Omega, \tag{4}$$

where function f_1 is determined by the equation

$$\partial_x[x f_1(x)] = \frac{1}{6}\int_0^\infty dq \frac{\sin(qx) - qx\cos(qx)}{qx}\frac{K_0(q)}{K_1(q)}$$

with the boundary condition $x f_1(x)|_{x\to\infty} = 0$ and

$$f_2(x) = \frac{1}{3}\int_0^\infty dq \frac{\sin(qx)}{qx}\frac{K_0(q)}{K_1(q)}.$$

With the obtained relations it is easy to show that for large values of the radial coordinate ($r \gg 1/k$) interesting from the cosmological point of view (see in what follows) we have

$$f_1(kr) = -\frac{1}{6k^2r^2}(2\ln(2kr) + 1), \quad f_2(kr) = \frac{\ln(2kr)}{3k^2r^2}. \tag{5}$$

Hence, as $r \to \infty$, the obtained metric goes over into a corresponding solution for the four-dimensional case. However, in contrast to the standard theory, the gravitational potential on the brane does not vanish and it behaves at large distances like $\ln(2kr)/k^2r^2$.

GEODESIC EQUATION. LENSING EFFECT

The results obtained enable the motion of a free falling massive particle or a photon to be studied in a static gravitational field of the form in question.

Writing the line element (4) as

$$\begin{gathered} ds^2 = -dt^2 A(r) + dr^2 + B(r)d\Omega_2, \\ A(r) = 1 - 8\pi G_4\eta^2 f_2(kr), \quad B(r) = r^2\left(1 - 8\pi G_4\eta^2(1 + f_1(kr))\right), \end{gathered} \tag{6}$$

we obtain the nonvanishing Kristoffel symbols

$$\Gamma^r_{tt} = \frac{1}{2}A', \;\; \Gamma^t_{rt} = \frac{1}{2}\frac{A'}{A}, \;\; \Gamma^r_{\theta\theta} = -\frac{1}{2}B', \;\; \Gamma^\theta_{\theta r} = \frac{1}{2}\frac{B'}{B},$$

$$\Gamma^r_{\varphi\varphi} = -\frac{1}{2}B'\sin^2\theta, \;\; \Gamma^\varphi_{\varphi r} = \frac{1}{2}\frac{B'}{B}, \;\; \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \;\; \Gamma^\varphi_{\varphi\theta} = \cot\theta,$$

where the primed symbol indicates the derivative with respect to r. Substituting them into the geodesic equations and considerating that, in virtue of the spherical symmetry of the field of the monopole, the orbit of the particle in question is in the equatorial plane $\theta = \pi/2$, we obtain the integrals of motion

$$\frac{dt}{dp}A = 1, \quad \frac{d\varphi}{dp}B = J, \quad \left(\frac{dr}{dp}\right)^2 - \frac{1}{A} + \frac{J^2}{B} = -E, \tag{7}$$

where p is the geodesic parameter chosen so that the first of the integrals of motion is equal to unity, and J plays the role of an angular momentum per unit energy of the particle.

It follows from the above relations that $ds^2 = -E\,dp^2$. Thus, $E > 0$ for massive particles and $E = 0$ for photons. Moreover, a particle's radius may reach the value of r if

$$\frac{1}{A(r)} \geq \frac{J^2}{B(r)} + E.$$

For the values of parameters assumed in this model ($k^{-1} < 1$ mm and $G_4\eta^2 \sim 10^{-6}$), in the scattering problem the situation is typical when the minimal distance between a particle and the center of a monopole satisfies the inequality $k^2 r_{\min}^2 \gg G_4\eta^2$. This is related to the fact that in the visible part of the Universe, a single monopole cat at most exist [5], and the distance between an observer and the monopole is of the order of $r_{\min}/G_4\eta^2$. In this case, unlimited trajectories are realized, and $(1-E)$ plays the role of the velocity squared of a particle at an infinite distance from the monopole.

It follows directly from (7) that

$$\left(\frac{dr}{d\varphi}\right)^2 = -B + \frac{B^2}{AJ^2} - \frac{B^2 E}{J^2}.$$

Hence, keeping in the right-hand side of the expression obtained the terms of the first order in $G_4\eta^2$, and also using (5) and (6), we obtain for large distances ($r \gg 1/k$)

$$\varphi(r \to \infty) = \pm\frac{\pi}{2} \pm \frac{1}{2}\frac{\partial}{\partial J}\frac{1}{J}\int_0^{\pi/2} d\varphi\left\{8\pi G_4\eta^2 J^2 - \frac{4\pi G_4\eta^2}{3k^2}\left[(1-E)\cos^2\varphi + 2\left(1+(1-E)\cos^2\varphi\right)\ln\left(\frac{2kJ}{\sqrt{1-E}\cos\varphi}\right)\right]\right\}$$

or

$$\varphi(r \to \infty) = \pm\frac{\pi}{2}(1+\varepsilon), \tag{8}$$

where

$$\varepsilon = 4\pi\eta^2 G_4 + \frac{2\pi\eta^2 G_4}{3k^2}\,\frac{(3-E)}{J^2}\left[\ln\left(\frac{4kJ}{\sqrt{1-E}}\right) - 1\right]. \tag{9}$$

Equation (8) describes the particle deflection by the angle $\pi\varepsilon$.

We consider in more detail the result obtained for the case when the particle in motion is a photon (the lensing effect). Let d be the distance between an observer and the monopole, and L be the distance between the monopole and a source. Then, being on the source–monopole line, the observer can see a circle with angular dimensions

$$\psi = 2\pi\varepsilon\frac{L}{L+d}. \tag{10}$$

Substituting (9) into (10) and taking into account that in the lowest order in the gravitational constant the angular momentum can be written in the form $J = \psi d/2$, we arrive at a nonlinear equation for the angle ψ. This can approximately be solved under the assumption that the distance between the monopole and the observer *a priori* meets the inequality $d \gg 1/kG_4\eta^2$. In this case, we obtain with adopted accuracy

$$\psi = 8\pi^2 G_4\eta^2\frac{L}{L+d} + \frac{L+d}{4\pi^2 G_4\eta^2 k^2 d^2 L}\left[\ln\left(\frac{16\pi^2 G_4\eta^2 kdL}{L+d}\right) - 1\right]. \tag{11}$$

It can also be shown that, being at a distance s from the source–monopole line and inside the cone with an apex angle $2\pi\varepsilon$ behind the monopole, the observer will see two images at an angle distance that is now determined by the relation

$$\psi' = \psi + s\left(\frac{1}{L+d} + \frac{1}{\pi\varepsilon L}\right)$$

and it is of the same order as that from (11).

The expressions obtained here differ from the analogous results of the standard four-dimensional cosmology [5] by the presence of the term that depends on the model parameter k. The corrections due to an additional dimension substantially depend on the distance and approximately behave like $1/d^2$.

CONCLUSION

Gravitational lensing provides a unique method of detecting exotic objects in the visible part of the Universe such as cosmic strings and a monopole. In this paper, we have considered a lensing effect in the case when a lens is produced by a global monopole embedded in the brane in the Randall–Sundrum universe of the second kind. We have demonstrated that, although the fields of the Standard Model and, in particular, the monopole itself, are localized on the brane, the presence of the fifth dimension has a certain effect on the character of motion of massive particles and propagation of light near a monopole. As in the case of four space–time dimensions, depending on the relative locations of the source of light, the monopole and the observer, either the monopole on the brane produces twin images, or the image of a point source is a circle. In both cases, the angular size of the image depends on the parameter k of the model, and it becomes more noticeable when the light beams pass closer to the monopole.

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