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BRIEF COMMUNICATIONS

THEORETICAL AND MATHEMATICAL PHYSICS

CLASSICAL SOLUTIONS OF EQUATIONS OF THE (2+1)-DIMENSIONAL GAUGE FIELD MODEL

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Basing upon the proposed method of numerical solutions of classical field equations, various modes of behavior of classical non-Abelian gauge fields have been studied in a (2+1)-dimensional space–time with consideration for the contribution of the Chern–Simons topological term and interaction with the Higgs fields.

1. The Yang–Mills fields, unlike free electromagnetic fields, have nontrivial topological structure due to their nonlinear character and related self-action [1–3]. These fields demonstrate rather complicated behavior already at the classical level. In papers [4, 5], it was shown that classical solutions of the Yang–Mills equations in the (3+1)-dimensional space demonstrate stochastic properties. In this work, following [6] where a different numerical approach was applied, we consider classical solutions of the Yang–Mills field equations interacting with Higgs fields in the (2+1)-dimensional space–time with consideration for the topological Chern–Simons term and we study possible stochastic modes of behavior of this system of fields.

2. We consider a (2+1)-dimensional Georgi–Glashow model with a topological Chern–Simons term in the $SU(2)$ group with field A_μ^a interacting with the Higgs field Φ^a, defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_\phi, \tag{1}$$

$$\mathcal{L}_g = -\frac{1}{4}{F_{\mu\nu}^a}^2 + \frac{1}{4}\theta\varepsilon^{\mu\nu\alpha}\left(F_{\mu\nu}^a A_\alpha^a - \frac{1}{3}g\varepsilon^{abc}A_\mu^a A_\nu^b A_\alpha^c\right), \tag{2}$$

$$\mathcal{L}_\phi = \frac{1}{2}(D_\mu\Phi^a)^2 + \frac{1}{2}m^2\Phi^a\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a)^2. \tag{3}$$

Here $\mu = 0,1,2$; $a = 1,2,3$; $D_\mu\Phi_a = \partial_\mu\Phi_a - ig\varepsilon_{abc}A_\mu^b\Phi^c$. Field equations have the form

$$D_\nu F^{\nu\mu a} + \frac{\theta}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}^a + g\varepsilon^{bac}\Phi^c D^\mu\Phi^b = 0, \tag{4}$$

$$D_\nu D^\nu\Phi^a - m^2\Phi^a + \lambda(\Phi^b\Phi^b)\Phi^a = 0. \tag{5}$$

The authors of [4, 5] have already demonstrated the possibility of obtaining stochastic solutions for the Yang–Mills fields in a (3+1)-dimensional model; therefore, we propose here a method for finding all possible solutions demonstrating stochastic behavior. In order to get particular solutions of systems of nonlinear equations (4), (5), we shall use the ansätze that have only some nonvanishing components depending on one space coordinate or time.

Our main goal is to find and study stochastic solutions; therefore, we have to construct a method that enables one to quickly obtain, from a new ansatz, equations for the functions that enter it and to analyze its solutions. It is known that most of the equations obtained in this way can only be solved numerically. To this end, we used the program in the Maple language. With the use of symbolic and numerical computing in combination, we obtained the system of equations immediately after setting an ansatz, and, should the system be consistent, we obtain solutions ready for analysis. This method helped us quickly study various types of ansätze.

To implement this method a special library for the Maple packet was produced that enabled equations of the Yang–Mills type to be written in a customary and convenient form. The customary form means that equations are written as they are usually written on paper, e.g., (4), (5), with the use of symbolic characters, operators, etc. Basing upon this, a program has been written that carries out absolutely all actions, necessary for finding and studying the solutions, by one instruction. The initial data for the program were: the ansatz (given in a customary form, e.g., A[a,mu]:=delta[a,mu]*f[a](t);), the space dimension and the model under study. In the present work, the two factors last mentioned were fixed (dimension 2+1, and the Yang–Mills and Higgs fields); however, they were also written in the general form. As a result, the program produced numerical solutions of the equations and the Poincaré surface. Where analytical solutions were possible, they were presented as well. Thus, the entire process of finding and analyzing the solutions was completely automated and carried out under one instruction that transferred the ansatz.

3. To demonstrate the method, in this work we restrict ourselves by the ansätze that depend on only one variable,

$$A_\mu^a = \delta_\mu^a f_a(x), \tag{6}$$

$$A_\mu^a = \varepsilon^{a\mu b} f_b(x), \tag{7}$$

$$A_\mu^a = |\varepsilon^{a\mu b}| f_b(x), \tag{8}$$

where x is understood as both space and time coordinate. It should be noted that all three ansätze give similar results. The same ansatz was everywhere used for Φ^a

$$\Phi^a = \phi_a(x). \tag{9}$$

Substituting (6) in (4), (5) and taking time t in place of variable x, a system of twelve differential equations of the second and the first order is obtained.

We consider solutions of the following type: $f_2 = f_3 = f$, $\phi_1 = \phi$, $\phi_2 = \phi_3 = 0$. Under these conditions, the system takes the form

$$\begin{aligned} & -f^3g^2 - \phi^2 g^2 f - \frac{1}{4}\theta^2 f - \frac{\partial^2}{\partial t^2} f = 0, \\ & \frac{\partial^2}{\partial t^2}\phi + 2f^2g^2\phi - m^2\phi + \lambda\phi^3 = 0, \\ & f_1 = -\frac{1}{2}\frac{\theta}{g}. \end{aligned} \tag{10}$$

Figure 1 (left picture) depicts the potential energy of this system

$$U = -\frac{1}{2}m^2\phi^2 + g^2f^2\phi^2 + \frac{1}{2}f^2\theta^2 + \frac{1}{2}f^4g^2 + \frac{1}{4}\lambda\phi^4. \tag{11}$$

Function (11) has three critical points: two minima with coordinates $\left(0, \pm\sqrt{m/\lambda}\right)$ and a saddle point with coordinates $(0,0)$.

Numerical analysis of system (10) demonstrated strong dependence of the field behavior on the initial conditions. Both quasiperiodic and stochastic modes were possible. For detailed study of trajectories, intersection points of trajectories with the plane $f = 0$, $f' > 0$ (the Poincaré surface) were found.

Figure 2 presents two such surfaces at different energy values. If the intersection points corresponding to one trajectory form a closed curve, the motion is quasiperiodic. If, on the contrary, the intersection

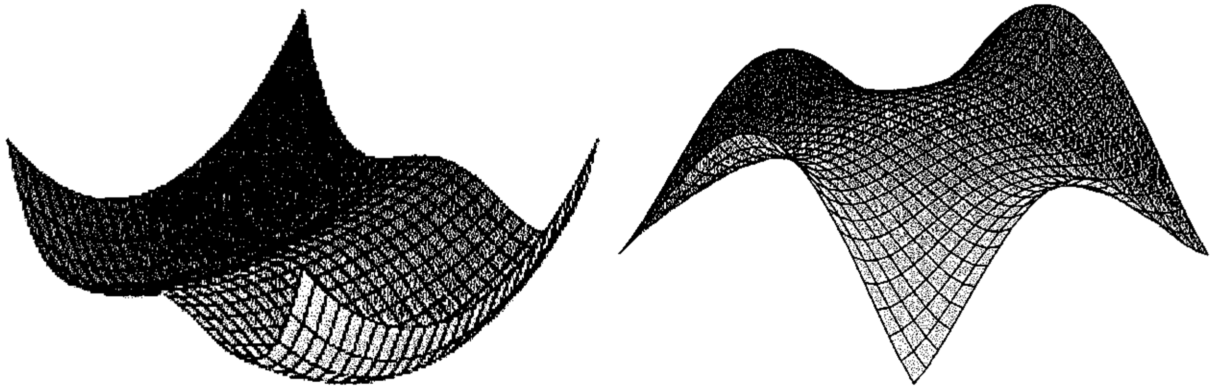

Fig. 1

Effective potential energy $U(f, \phi)$ for $g = 1$, $\theta = 1$, $\mu = 1$, $\lambda = 2$; at the left: $x = t$; at the right: $x = y$.

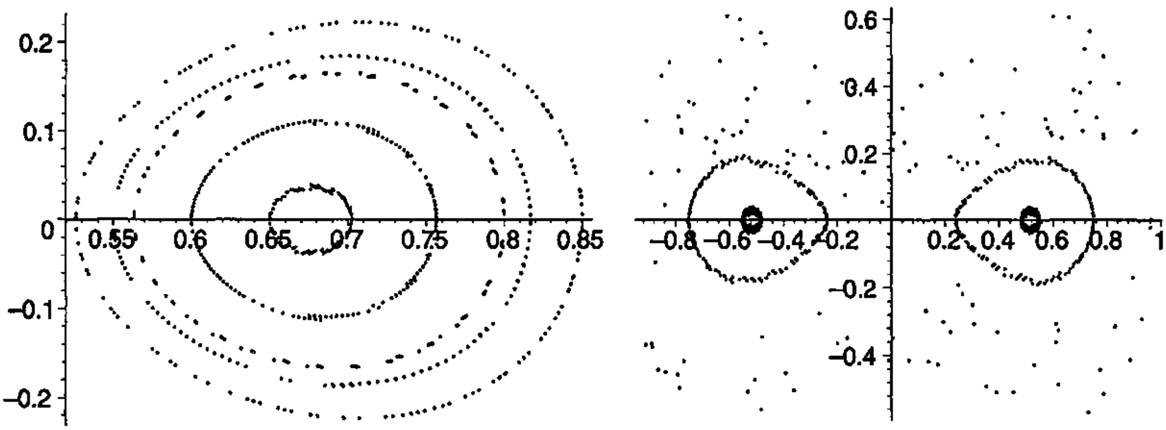

Fig. 2

Poincaré surface $\mathcal{E} = -0.1$ (at the left) and $\mathcal{E} = 0.1$ (at the right) ($g = 1$, $\theta = 1$, $\mu = 1$, $\lambda = 2$).

points corresponding to one trajectory fill a certain region, the motion has stochastic character. The curve at the left in Fig. 2 corresponds to negative energy lying below the saddle point. As it is clear from the figure, trajectories are quasiperiodic. For higher energy, a quite different situation arises illustrated by the curve at the right of Fig. 2, i.e., the whole space is chaotically covered, and there are closed curves only in small regions. Thus, stochastic motion dominates in this region. For even higher energies, the situation again changes. The motion becomes more and more stable, as it is quite clear from Fig. 3.

It follows from the above numerical calculation that in the region of the saddle point, the character of motion changes. Trajectories become more unstable, and for energy values closer to the saddle point, motion becomes more chaotic.

4. Substituting (6) in (4), (5) and taking space coordinate y in place of variable x, we obtain a system of twelve differential equations of the second and the first order with respect to y. We consider solutions of the following type: $f_1 = f$, $f_2 = if$, $\phi_1 = 0$, $\phi_2 = \phi_3 = \phi$. In this case, the potential energy of the system takes the form (at the left of Fig. 1)

$$U = \frac{1}{2}f^4 g^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - \frac{1}{4}\theta^2 f^2 - g^2 f^2 \phi^2. \tag{12}$$

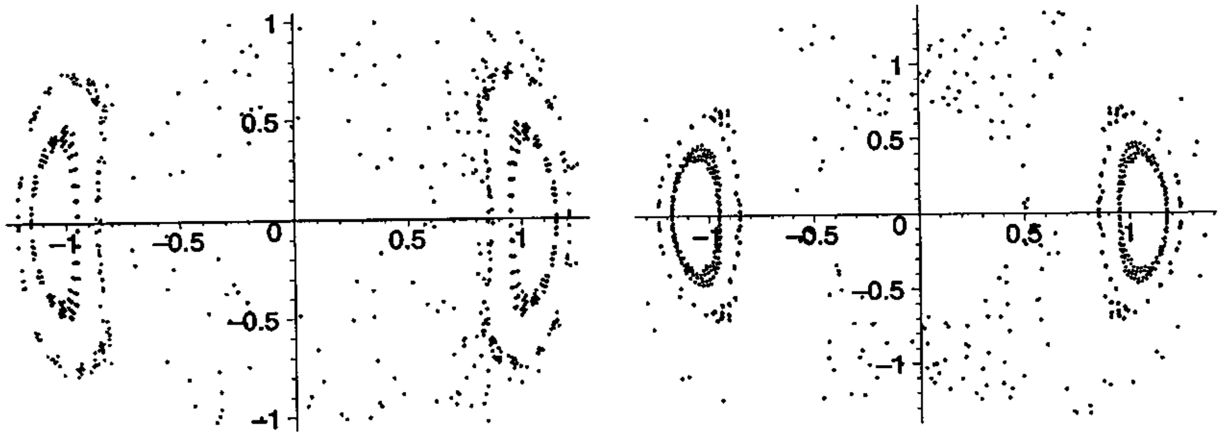

Fig. 3

Poincaré surface $\mathcal{E} = 0.5$ (at the left) and $\mathcal{E} = 0.8$ (at the right) ($g = 1$, $\theta = 1$, $\mu = 1$, $\lambda = 2$).

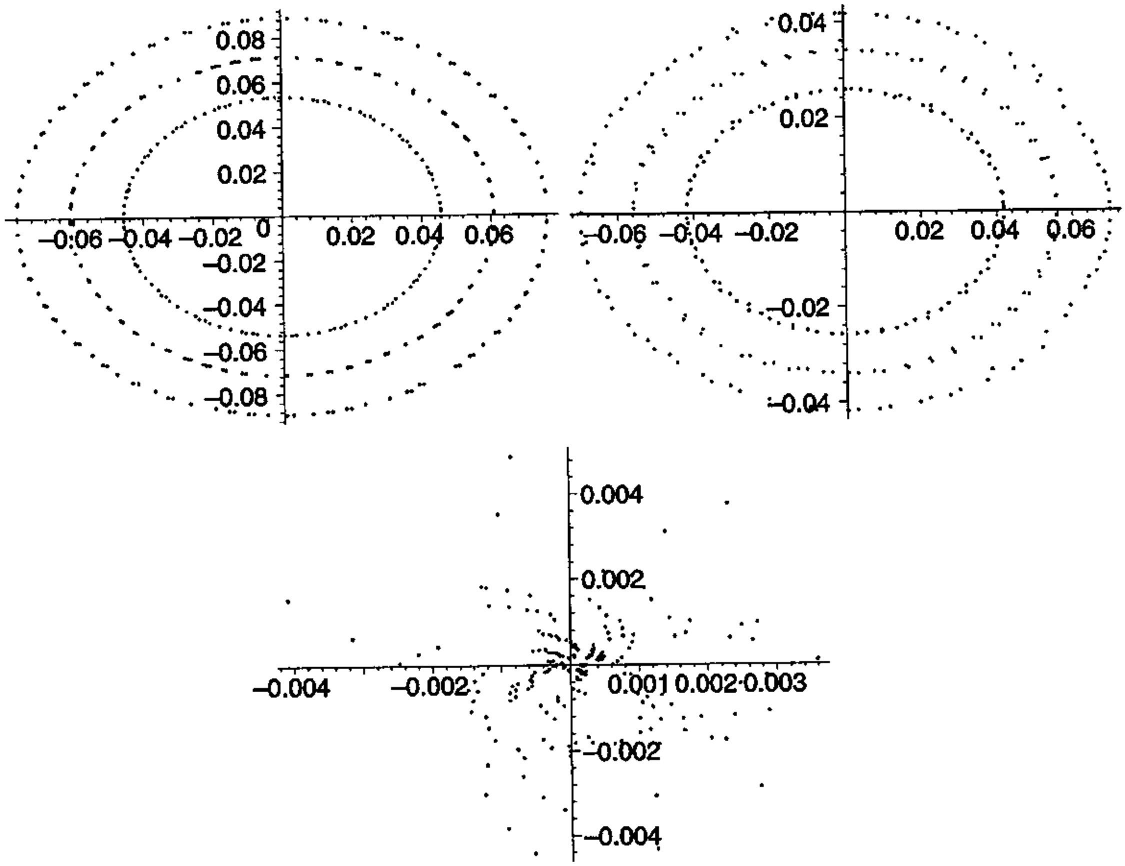

Fig. 4

Poincaré surface $\mathcal{E} = -0.2$ (left), $\mathcal{E} = 0.2$ (right), $\mathcal{E} = -0.125$ (bottom) ($g = 1$, $\theta = 1$, $\mu = 1$, $\lambda = 2$).

Numerical calculations of the corresponding system of equations

$$\begin{aligned}
&\phi^2 g^2 f + \frac{1}{4}\theta^2 f - f^3 g^2 - \frac{\partial^2}{\partial y^2} f = 0, \\
&-2f^2 g^2 \phi - \frac{\partial^2}{\partial y^2}\phi - m^2\phi + \lambda\phi^3 = 0, \\
&f_3 = \frac{-i}{2}\frac{\theta}{g}
\end{aligned} \tag{13}$$

have been conducted under various initial conditions. The obtained results resemble those of item 3. In order to make a detailed study of the solutions, let us construct a Poincaré surface. It is clear from Fig. 4 that, as in item 3 above, there exists a region of energies, where the motion has stochastic character. The degree of chaotic character of trajectories, again as in item 3 above, depends on how close the energy is to the saddle point.

5. The main physical conclusion of papers by Matinyan and Savvidy [4–10] is that the system of Yang–Mills fields in the (3+1)-dimensional space–time cannot be exactly solved since, otherwise, the trajectories should have a regular, and not chaotic form. In this paper, we have confirmed this conclusion for the (2+1)-dimensional gauge theory with a Chern–Simons topological term and interaction with the Higgs field, and also the conclusion of [6] obtained with the help of an alternative numerical approach. The method developed in solving the problems posed in the present work can be useful in quickly finding and analyzing nontrivial solutions of systems of nonlinear equations.

Further development of the method proposed is of interest for searching for solutions of this kind for other ansätze, as well as for other models.

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