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SPECTRAL PROBLEM FOR A WAVEGUIDE WITH IMPEDANCE BOUNDARY CONDITIONS

A. N. Bogolyubov, M. D. Malykh, and Yu. V. Mukhartova

Normal waves in a hollow cylindric waveguide Ω with a smooth-boundary cross section S have been considered. The Shchukin–Leontovich boundary conditions at the waveguide boundary were specified. It has been demonstrated that the propagation constant cannot be purely real in this system.

One of the ways to consider for electromagnetic field energy absorption in well-conducting media (metals) is the use of the equivalent Shchukin–Leontovich boundary conditions [1, 2]

$$[\mathbf{n}, \mathbf{E}] = \varsigma[\mathbf{n}, [\mathbf{n}, \mathbf{H}]], \quad \varsigma = \sqrt{\frac{\omega\mu}{2\sigma_0}}(i-1), \tag{1}$$

where σ_0 is the conductivity of metal at constant current. Equation (1) is written for monochromatic fields or for the Fourier components in the case of an arbitrary field.

We consider a problem of normal waves in a regular hollow cylindric waveguide

$$\Omega = \{(x, y) \in S;\ z \in R\} \tag{2}$$

under the condition that the boundary of the region S is sufficiently smooth and the Shchukin–Leontovich boundary conditions are met at $\partial\Omega$,

$$\begin{cases} \operatorname{curl} \mathbf{H} = -i\omega\mathbf{E}, \\ \operatorname{curl} \mathbf{E} = i\omega\mathbf{H}, \\ [\mathbf{n}, \mathbf{E}]|_{\partial\Omega} = \varsigma[\mathbf{n}, [\mathbf{n}, \mathbf{H}]]. \end{cases} \tag{3}$$

Problem (3) is well studied for $\varsigma = 0$, and its eigenwave system is complete. However, at $\varsigma \neq 0$ this condition is not met in the general case. In what follows, we shall consider only the case $\varsigma \neq 0$. Since the equations for the fields $\mathbf{E}$ and $\mathbf{H}$ are homogeneous, the solution of (3) can be sought in the form

$$\begin{aligned} \mathbf{E} &= \operatorname{grad}(\operatorname{div} \mathbf{\Pi}^e) + \omega^2\mathbf{\Pi}^e - i\omega \operatorname{curl} \mathbf{\Pi}^m, \\ \mathbf{H} &= i\omega \operatorname{curl} \mathbf{\Pi}^e + \operatorname{grad}(\operatorname{div} \mathbf{\Pi}^m) + \omega^2\mathbf{\Pi}^m, \end{aligned} \tag{4}$$

where the electric and magnetic Hertz vectors are directed along the waveguide axis

$$\mathbf{\Pi}^e = \varphi(x, y)e^{i\gamma z} \cdot \mathbf{e}_z, \quad \mathbf{\Pi}^m = \psi(x, y)e^{i\gamma z} \cdot \mathbf{e}_z. \tag{5}$$

Let vector $\mathbf{n}$ be a normal to the boundary of the waveguide and $\boldsymbol{\tau} = [\mathbf{e}_z, \mathbf{n}] = (-n_y, n_x, 0)$ be a vector tangential to the boundary. In this case problem (3) is reduced to the system [2]

$$\Delta_2\varphi + (\omega^2 - \gamma^2)\,\varphi = 0, \tag{6}$$

$$\Delta_2\psi + (\omega^2 - \gamma^2)\,\psi = 0, \tag{7}$$

$$i\gamma\frac{\partial\varphi}{\partial\tau} + i\omega\frac{\partial\psi}{\partial\mathbf{n}} + \varsigma\,(\omega^2 - \gamma^2)\,\psi|_{\partial\Omega} = 0, \tag{8}$$

$$-(\omega^2 - \gamma^2)\,\varphi + \varsigma\left(i\gamma\frac{\partial\psi}{\partial\tau} - i\omega\frac{\partial\varphi}{\partial\mathbf{n}}\right)\Big|_{\partial\Omega} = 0. \tag{9}$$

We introduce the following notation:

$$u = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad I_1 = \begin{pmatrix} -\varsigma & 0 \\ 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & \varsigma \\ 1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} -1 & 0 \\ 0 & \varsigma \end{pmatrix}. \tag{10}$$

Function u satisfies the problem

$$\Delta_2 u + \left(\omega^2 - \gamma^2\right) u = 0, \tag{11}$$

$$i\omega I_1 \frac{\partial}{\partial \mathbf{n}} u + i\gamma I_2 \frac{\partial}{\partial \tau} u + I_3 \left(\omega^2 - \gamma^2\right) u\big|_{\partial\Omega} = 0. \tag{12}$$

The specific feature of this problem is that its propagation constant γ is a pure imaginary quantity. Let us demonstrate that this is true.

The derivative of function u along the normal to the boundary of region S has the form

$$\frac{\partial u}{\partial \mathbf{n}} = -\frac{\gamma}{\omega} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial u}{\partial \tau} + \frac{i}{\omega} \left(\omega^2 - \gamma^2\right) \begin{pmatrix} 1/\varsigma & 0 \\ 0 & \varsigma \end{pmatrix} u. \tag{13}$$

Multiplying equation (11) formally by the vector function $v(M) = (v_1(M), v_2(M))^{\mathrm{T}}$, where $v_1(M)$ and $v_2(M)$ belong to the space $W_2^1(S)$, we integrate the result over the waveguide cross section S. Considering the boundary conditions, we obtain the following identity:

$$-\int\limits_S \left\{\nabla v^{\mathrm{T}} \nabla u + v^{\mathrm{T}} u\right\} ds + \left(\omega^2 - \gamma^2 + 1\right) \int\limits_S v^{\mathrm{T}} u \, ds + \frac{i}{\omega}\left(\omega^2 - \gamma^2\right) \oint\limits_{\partial S} \left\{\frac{1}{\varsigma}\bar{v}_1 u_1 + \varsigma \bar{v}_2 u_2\right\} d\tau - \frac{\gamma}{\omega} \oint\limits_{\partial S} \left\{-\bar{v}_1 \frac{\partial u_2}{\partial \tau} + \bar{v}_2 \frac{\partial u_1}{\partial \tau}\right\} d\tau = 0. \tag{14}$$

Consider a function $u(\gamma, M)$ with components belonging to the Sobolev space $W_2^1(S)$, that satisfies identity (14) for any function $v(M)$, such that $v_1(M) \in W_2^1(S)$ and $v_2(M) \in W_2^1(S)$ [3]. This function will be called the generalized solution of problem (11), (12). Let a certain quantity $\gamma = \gamma_1 + i\gamma_2$, complex-valued in the general case, and a function $w(\gamma, M) = (w_1(\gamma, M), w_2(\gamma, M))^{\mathrm{T}}$ satisfy problem (11), (12). Moreover, γ should be a root of the quadratic equation

$$\frac{1}{\gamma^2}\|w\|^2_{W_2^1(S)} + \int\limits_S |w|^2 \, ds - \frac{1+\omega^2}{\gamma^2} \int\limits_S |w|^2 \, ds - \frac{i}{\omega\gamma^2}\left(\omega^2 - \gamma^2\right) \oint\limits_{\partial S} \left\{\frac{1}{\varsigma}|w_1|^2 + \varsigma |w_2|^2\right\} d\tau + \frac{1}{\omega\gamma} \oint\limits_{\partial S} \left\{-\bar{w}_1 \frac{\partial w_2}{\partial \tau} + \bar{w}_2 \frac{\partial w_1}{\partial \tau}\right\} d\tau = 0. \tag{15}$$

Transform the expression in the last integrand of equation (15):

$$-\bar{w}_1 \frac{\partial w_2}{\partial \tau} + \bar{w}_2 \frac{\partial w_1}{\partial \tau} = -\operatorname{Re} w_1 \cdot \operatorname{Re} \frac{\partial w_2}{\partial \tau} - \operatorname{Im} w_1 \cdot \operatorname{Im} \frac{\partial w_2}{\partial \tau} + \operatorname{Re} w_2 \cdot \operatorname{Re} \frac{\partial w_1}{\partial \tau} + \operatorname{Im} w_2 \cdot \operatorname{Im} \frac{\partial w_1}{\partial \tau} + i \frac{\partial}{\partial \tau}\{\operatorname{Re} w_2 \cdot \operatorname{Im} w_1 - \operatorname{Re} w_1 \cdot \operatorname{Im} w_2\}. \tag{16}$$

Considering the imaginary part of (15), with allowance for

$$\varsigma = \frac{|\varsigma|}{\sqrt{2}}(i-1) \quad \text{and} \quad \frac{1}{\varsigma} = -\frac{1}{|\varsigma|\sqrt{2}}(i+1),$$

the following equation is obtained:

$$-\frac{2\gamma_1\gamma_2}{|\gamma^2|^2}\|w\|^2_{W_2^1(S)}+\frac{2\gamma_1\gamma_2(1+\omega^2)}{|\gamma^2|^2}\int\limits_S|w|^2\,ds+\frac{(\gamma_1^2-\gamma_2^2)\,\omega}{|\gamma^2|^2}\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2+\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}d\tau$$
$$+\frac{2\gamma_1\gamma_2\omega}{|\gamma^2|^2}\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2-\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}d\tau-\frac{1}{\omega}\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2+\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}d\tau$$
$$+\frac{\gamma_1}{\omega|\gamma|^2}\oint\limits_{\partial S}\frac{\partial}{\partial\tau}\{\mathrm{Re}\,w_2\cdot\mathrm{Im}\,w_1-\mathrm{Re}\,w_1\cdot\mathrm{Im}\,w_2\}\,d\tau-\frac{\gamma_2}{\omega|\gamma|^2}\oint\limits_{\partial S}\mathrm{Re}\left\{-\bar{w}_1\frac{\partial w_2}{\partial\tau}+\bar{w}_2\frac{\partial w_1}{\partial\tau}\right\}d\tau=0. \quad (17)$$

Assume γ is real, i. e., let $\gamma_2 = 0$. In this case, it follows from equation (17) that the following condition should be met:

$$\frac{\omega}{\gamma_1^2}\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2+\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}d\tau-\frac{1}{\omega}\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2+\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}d\tau=0 \quad (18)$$

since

$$\oint\limits_{\partial S}\frac{\partial}{\partial\tau}\{\mathrm{Re}\,w_2\cdot\mathrm{Im}\,w_1-\mathrm{Re}\,w_1\cdot\mathrm{Im}\,w_2\}\,d\tau=0. \quad (19)$$

Therefore, either $\gamma_1^2=\omega^2$, or

$$\oint\limits_{\partial S}\left\{\frac{1}{|\varsigma|\sqrt{2}}|w_1|^2+\frac{|\varsigma|}{\sqrt{2}}|w_2|^2\right\}dl=0.$$

The expressions in the integrand are nonnegative, and hence, the integral can be equal to zero only if $w_1|_{\partial S}=w_2|_{\partial S}=0$ for ς being modulo arbitrary. Moreover, it follows from condition (13) that $\frac{\partial w}{\partial n}|_{\partial S}=0$, as well. Therefore, if γ is a real quantity, the corresponding function w must satisfy the problem

$$\Delta_2 w+\left(\omega^2-\gamma^2\right)w=0, \quad (20)$$
$$w|_{\partial S}=0, \quad (21)$$
$$\left.\frac{\partial w}{\partial n}\right|_{\partial S}=0, \quad (22)$$

having only trivial solutions.

REFERENCES

1. A.S. Il'inskii and G.Ya. Slepyan, *Oscillations and Waves in Electromagnetic Systems with Losses* (in Russian), Moscow, 1983.
2. A.S. Il'inskii, V.V. Kravtsov, and A.G. Sveshnikov, *Mathematical Models in Electrodynamics* (in Russian), Moscow, 1991.
3. O.A. Ladyzhenskaya, *Boundary Problems in Mathematical Physics* (in Russian), Moscow, 1973.

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