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The Mixed Finite Element Method as Applied to Waveguide Diffraction Problems

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Abstract—A mixed finite element algorithm for solving the vector problem of diffraction by an inhomogeneity in a waveguide is constructed and implemented. Partial radiation conditions are used to proceed to a bounded domain.

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INTRODUCTION

In [1–3], the scalar problem of diffraction by an inhomogeneity in a waveguide was solved by applying the finite element method and partial radiation conditions were used to proceed to a bounded solution domain. The scalar setting involves the basic points associated with projection-grid methods as applied to waveguide diffraction problems, yet it does not exhibit properties that sharply reduce the effectiveness of numerical algorithms, namely, it is free of spurious unphysical solutions that are typical of vector settings of waveguide diffraction problems [4].

An important point in this case is that the operator $\mathrm{curl}\,\varepsilon^{-1}\mathrm{curl}$ has a zero eigenvalue of infinite multiplicity. When we use the finite element method with classical Lagrangian finite elements, the electromagnetic field is approximated by a set of continuous functions. After discretizing the problem, the zero eigenvalue turns into a family of nonzero ones with their eigenfunctions giving spurious solutions. Note that, in the case of Lagrangian finite elements, spurious solutions can also be generated by nonzero eigenvalues [5].

Nonphysical solutions can be coped with by various methods, which can be divided into two large classes: a posteriori and a priori [4, 5]. A posteriori methods represent various techniques for detecting and eliminating spurious solutions. They are rather laborious and frequently not very effective. The usual method is based on the verification of a conservative equation, which is violated by spurious solutions.

In the case of a priori methods, the original problem is formulated so as to eliminate the generation of spurious solutions or, at least, certain types of such solutions. Several major approaches are available. In one of them, conservative equations are explicitly included in the system of equations to be solved. An example is the previously popular penalty functional method, in which the required conditions are taken into account via undetermined Lagrange multipliers, thus leading to a nonlinear minimization problem. As a rule, the penalty parameter is chosen from experiments. Due to the approximate choice of the penalty parameter, nonphysical solutions are poorly eliminated, which is aggravated by the considerable complexity of the method. As a result, this technique is now rarely used. Another approach is that in which a conservative equation is used to eliminate the longitudinal component of the magnetic field and the problem is formulated in transverse coordinates. In this case, the main shortcoming is the filling in of the matrices associated with the finite element method, which sharply reduces the efficiency of this approach.

In our view the preferable approach is that in which the zero eigenvalue is approximated precisely. The zero eigenvalue is then discarded at the end of the computations or is algebraically eliminated from the discrete problem. In this context, a promising technique is based on mixed finite elements [6], since the zero eigenvalue can be precisely calculated by the corresponding algorithms. Note also that the approximate solution u_h to the original problem belongs to a finite-dimensional piecewise polynomial space. In the case of Lagrangian finite elements, this can be achieved only if u_h is continuous. Therefore, in the latter case, a solution to the discrete problem has to be sought among continuous functions, while the solution to the original differential problem may be discontinuous. For example, given the eigenvalue problem for magnetic field components with the eigenvalue being the squared wave number, its solution belongs to the set of vector functions satisfying $\mathrm{curl}\,\mathbf{H} \in L_2$, i.e., functions admitting a discontinuous normal field component, where the normal is drawn to the edge of a finite element.

In this paper, the diffraction of an electromagnetic wave by an inhomogeneity in a waveguide is computed using mixed finite elements.

STATEMENT OF THE PROBLEM

The vector problem of scattering electromagnetic waves by a local inhomogeneity in a waveguide with no absorption (two-dimensional case) is formulated as follows. The waveguide is infinite, and its walls are assumed to be perfectly conducting. We introduce a Cartesian coordinate system such that the z axis is aligned with the waveguide axis and is perpendicular to the x axis. In connection with the problem under study, consider the equation

$$\begin{gathered}\text{curl}\,\varepsilon^{-1}\text{curl}\mathbf{H} - k^2\mathbf{H} = 0 \\ \text{div}\mathbf{H} = 0\end{gathered} \tag{1}$$

in $\Omega = \{z \in (-\infty, \infty); x \in [0, 1]\}$ with homogeneous boundary conditions set on the lateral surface of the waveguide (at $x = 0$ and $x = 1$):

$$[\text{curl}\mathbf{H} \times \mathbf{n}] = 0, \tag{2}$$

where $\mathbf{H}$ is a field with components H_z and H_x ($\mathbf{H} = \{H_z, H_x\}$), $\varepsilon(z)$ is the permittivity given by

$$\varepsilon(z) = \begin{cases} 1, & z < z_1, \quad z > z_2 \\ \varepsilon \geq 1, & z_1 < z < z_2, \end{cases} \tag{3}$$

and k is the wave number.

The inhomogeneity occupies the domain $D = \{z \in (z_1, z_2); x \in [0, 1]\}$ bounded by the cross sections $S_1 = \{z = z_1; x \in [0, 1]\}$ and $S_2 = \{z = z_2; x \in [0, 1]\}$, on which we set the partial radiation conditions [7].

A solution is sought in the form $\mathbf{H} = \text{curl}\chi$, where $\text{curl}\chi = \frac{\partial\chi}{\partial x}\mathbf{e}_z - \frac{\partial\chi}{\partial z}\mathbf{e}_x$.

The desired solution is expanded in the eigenfunctions for χ, which are determined by solving the auxiliary problem

$$\Delta\chi + k^2\chi = 0 \text{ in } D = \{z \in (z_1, z_2); x \in [0, 1]\}.$$

$$(\text{curl}\chi e_x)\big|_{\substack{x=0\\x=1}} = 0.$$

Its solution is written as

$$\chi_n(z, x) = \sin(\pi n x)e^{\pm i\gamma_n z}. \tag{4}$$

The general solution to the auxiliary problem has the form

$$\chi = \sum_{n=1}^{\infty} T_n \sin(\pi n x)e^{i\gamma_n z} + \sum_{n=1}^{\infty} R_n \sin(\pi n x)e^{-i\gamma_n z}, \tag{5}$$

where T_n and R_n are the mode transmission and reflection coefficients. To be definite, suppose that the incident wave is the first waveguide mode with amplitude C_1; i.e., $\chi_1 = C_1\sin(\pi x)e^{i\gamma_1 z}$. In the general case, the incident wave can be expanded in a Fourier series in terms of the waveguide modes $\chi_n(z, x)$. Then the desired field can be obtained as the sum of the solutions for all the modes with corresponding coefficients.

The resulting general solution (5) is used to derive the radiation conditions for χ. Taking into account these conditions, we can write

$$\chi = C_1\sin(\pi x)e^{i\gamma_1 z} + \sum_{n=1}^{\infty} R_n \sin(\pi n x)e^{-i\gamma_n z} \tag{6}$$

for $z \leq z_1$ and

$$\chi = \sum_{n=1}^{\infty} T_n \sin(\pi n x)e^{i\gamma_n z} \tag{7}$$

for $z \geq z_2$.

To apply the finite element method, we need a generalized statement of problem (1)–(3), (6), (7). Assume that H is a solution to this problem. Multiplying Eq. (1) by a sufficiently smooth vector function H^* and integrating the result over D, after making rearrangements and using the expressions for C_1, R_n, and T_n, we obtain the following generalized statement of the problem:

$$\begin{aligned}&\int_D \varepsilon^{-1}\text{curl}H_\perp \text{curl}H_\perp^* dzdx - \int_D k^2(H^*, H)dzdx \\ &- 2i\varepsilon^{-1}\sum_{n=1}^{\infty}\left(\gamma_n + \frac{\pi^2 n^2}{\gamma_n}\right)(H_x, \sin\pi n x)_{S_1}(H_x^*, \sin\pi n x)_{S_1} \\ &- 2i\varepsilon^{-1}\sum_{n=1}^{\infty}\left(\gamma_n + \frac{\pi^2 n^2}{\gamma_n}\right)(H_x, \sin\pi n x)_{S_2}(H_x^*, \sin\pi n x)_{S_2} \\ &= -2B\varepsilon^{-1}(\gamma_1^2 + \pi^2)e^{i\gamma_1 z_1}(H_x^*, \sin\pi x)_{S_1}.\end{aligned} \tag{8}$$

SOLUTION ALGORITHM

The problem under study is solved by the mixed finite element method. The field H_z is approximated by $N_j(x)p_{i,\,i+1}(z)$, and H_x is approximated by $N_i(z)p_{j,\,j+1}(x)$, where $N_i(x)$ is equal to 1 at the ith node, is piecewise linear on the intervals $[(i-1)h_x, ih_x]$ and $[ih_x, (i+1)h_x]$, and vanishes outside the interval $[(i-1)h_x, (i+1)h_x]$;

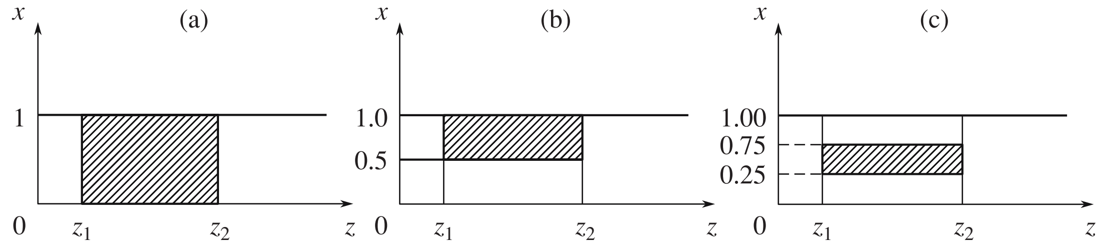

Fig. 1. Waveguides with (a) inhomogeneity of the plug type and with inhomogeneity placed in (b) the upper half of the waveguide and (c) the middle of the waveguide.

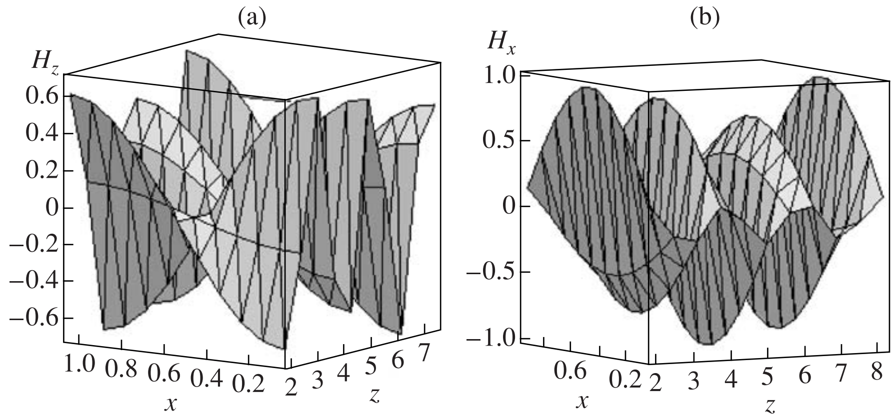

Fig. 2. Field distribution in the waveguide with a plug inhomogeneity ($\varepsilon = 2$): (a) the component H_z of **H** and (b) the component H_x of **H**.

and $p_{i,\,i+1}(x)$ is equal to 1 on $[ih_x, (i+1)h_x]$ and vanishes outside this interval:

$$\begin{pmatrix} H_z \\ H_x \end{pmatrix} = \begin{pmatrix} \sum\limits_{i,\,j} H_{z_{ij}} N_j(x) p_{i,\,i+1}(z) \\ \sum\limits_{i,\,j} H_{x_{ij}} N_i(z) p_{j,\,j+1}(x) \end{pmatrix}.$$

For convenience, we use the notation $N_j(x)p_{i,\,i+1}(z) = N_i(z,\,x)$ for H_z and $N_i(z)p_{j,\,j+1}(x) = M_j(z,\,x)$ for H_x, where $i \in [1, N_x(N_z+1)]$ and $j \in [1, N_z(N_x+1)]$. Then H_z and H_x represented as $H_z = \sum H_{zi} N_i(z, x)$ and $H_x = \sum H_{xj} M_j(z, x)$ are substituted into the variational statement of problem (8).

Any function satisfying the boundary conditions can be used as H^*. Successively replacing H^* with the finite elements N_i (for H_z^*) and M_j (for H_x^*), we obtain a set of linear equations for the coefficients H_{zi} and H_{xj}. Thus, the problem is reduced to a set of linear algebraic equations with a coefficient matrix A and a right-hand side column B. Here, A is a square matrix and its dimension is equal to the total number of the functions N_i and M_j; e.g., for a rectangular domain with an $N_z \times N_x$ grid, the dimension of A is $N_x(N_z + 1) + N_z(N_x + 1)$.

Note that A is a symmetric, sparse matrix. This circumstance considerably simplifies both the construction of the matrix and the solution of the resulting system of linear algebraic equations $Au = B$.

RESULTS

The problem was solved for several types of inhomogeneity. The incident wave was specified as the first eigenmode $\chi_1 = C_1\sin(\pi x)e^{i\gamma_1 z}$, which propagated in the positive z direction. The real parts of the components H_z and H_x of H were depicted on an axis perpendicular to the plane (z, x).

Figure 2 shows the field distribution in the waveguide for a plug-type inhomogeneity specified as $\varepsilon = 2$ for $x \in (0, 1)$ and $z \in (z_1, z_2)$ (see Fig. 1a). It can be seen that, in the absence of absorption, the amplitude virtually does not vary. The effect of absorption on the wave amplitude can be clearly seen in Fig. 3 (in this case, $\varepsilon = 2 + i$). Strong absorption ($\mathrm{Im}\varepsilon/\mathrm{Re}\varepsilon = 1/2$)

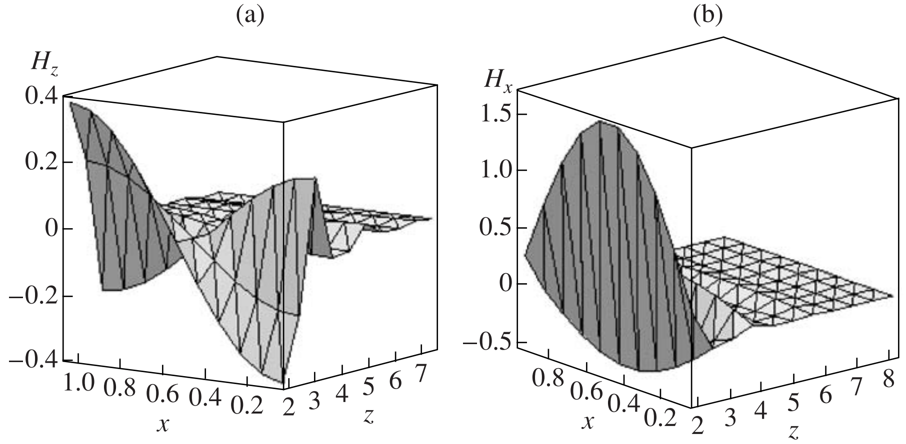

Fig. 3. Field distribution in the waveguide with a plug inhomogeneity ($\varepsilon = 2 + i$): (a) the component H_z of **H** and (b) the component H_x of **H**.

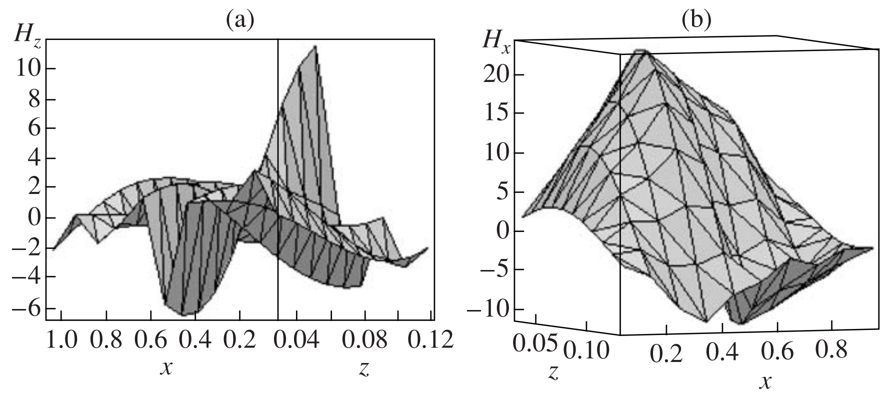

Fig. 4. Waveguide field distribution for a inhomogeneity ($\varepsilon = 2 + i$) placed in the upper half of the waveguide: (a) the component H_z of **H** and (b) the component H_x of **H**.

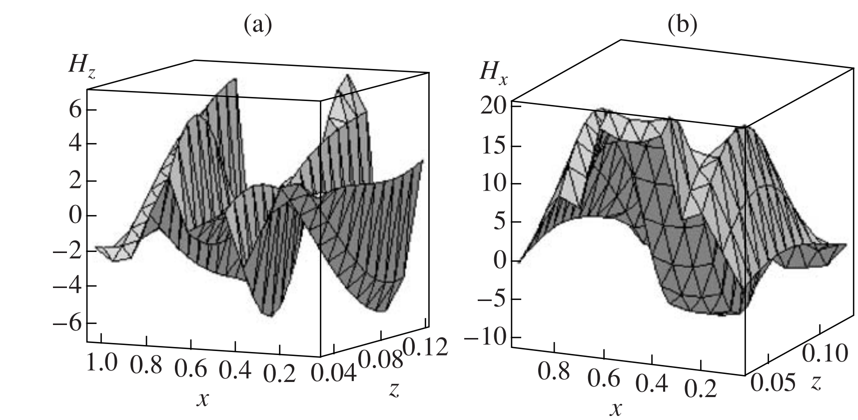

Fig. 5. Waveguide field distribution for a inhomogeneity ($\varepsilon = 2 + i$) placed in the middle of the waveguide: (a) the component H_z of **H** and (b) the component H_x of **H**.

leads to a considerably reduced field intensity (here, $\mathrm{Im}\,\varepsilon$ and $\mathrm{Re}\,\varepsilon$ denote the imaginary and real parts of the permittivity, respectively).

Figure 4 displays the waveguide field distribution for an inhomogeneity occupying the upper half of the waveguide: $\varepsilon = 2 + i$ for $x \in (1/2, 1)$ and $z \in (z_1, z_2)$ (see Fig. 1b). For this asymmetric insertion, the field is pulled into the domain of higher optical density. Due to the absorption, the field amplitude in the inhomogeneity area is relatively leveled.

Figure 5 shows the waveguide field distribution for an inhomogeneity placed in the middle of the waveguide, $\varepsilon = 2 + i$ (see Fig. 1c). It can be clearly seen that the wave field is concentrated in the central domain.

CONCLUSIONS

The results presented show that mixed finite elements can be used to design effective numerical algorithms that prevent the generation of spurious unphysical solutions.

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