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Some Qualitative Properties of the Van der Pol Equation Inferred from Its Singular Set

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Abstract—It is shown that the singular set of an extended "inverse" integral curve $x(y)$ of the Van der Pol equation is covered with local extrema of $x(y)$ that are stable with respect to small perturbations in the equation. As a consequence, the qualitative behavior of $x(y)$ can be determined and some of its important properties can be understood.

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INTRODUCTION

For convenience, the Van der Pol equation [1] is written as the system

$$\dot{y} = -x + \varepsilon y(1-x^2), \quad \dot{x} = y, \quad \varepsilon > 0. \qquad (1)$$

This serves as a major model of self-sustained oscillations in mechanical and electronic systems [2, 3], biology and biochemistry [4, 5], and other applications. This equation has been intensively studied.

To analyze the behavior of an integral curve $x(y)$ of the Van der Pol equation, we use a method in which its solution is extended to its singular set by using extensions of the local first integrals to this set (see [6–8]). It is shown that, for the extended curve, the singular set is entirely covered with local extrema of $x(y)$ that are stable with respect to small perturbations in the equation.

To determine the singular set, we consider the system

$$\dot{y} = P(x, y), \quad \dot{x} = y, \quad P \in C^{1,1}, \qquad (2)$$
$$(x, y) \in M \subseteq \mathbb{R}^2, \quad t \in T \subseteq \mathbb{R},$$

assuming that a trajectory $(\varphi(t), \dot{\varphi}(t)) \in M$ exists.

It follows from (2) that

$$dx/dy = y/P(x, y) \in C^{1,1}, \quad P(x, y) \neq 0. \qquad (3)$$

The set $S = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 0, y \neq 0\}$ is called a singular set of the second type, and its points are referred to as S-singular points. Assume that S is of Lebesgue measure zero on $\mathbb{R}^2$. Let the first integral $I(x, y)$ of Eq. (2) be extendable in $C^{1,1}$ to a neighborhood O of a singular point. The curve $[x(t), y(t)]$ is the extension of the trajectory $(\varphi(t), \dot{\varphi}(t))$ to $M \cup O$ by using $I(x, y)$ if $x(t) = \varphi(t)$ and $y(t) = \dot{\varphi}(t)$ for $t \in T$ and $I(x(t), y(t)) \equiv C$ in O (where C is a constant). As a consequence, the integral curves $x(y)$ and $y(x)$ are extended to $M \cup O$.

Define $\Xi_c = \{(x, y) \in \mathbb{R}^2 : I(x, y) = C\}$. An integral curve reaches S only if $\Xi_c \cap S \neq \emptyset$.

For convenience, we say that (x_0, y_0) is a point of local extremum of $x(y)$: $\mathbb{R} \longrightarrow \mathbb{R}$ if $x(y)$ has a local extremum at $y = y_0$ and $x_0 = x(y_0)$.

Define $l_\pm = \lim \frac{dx}{dy}(x, y)$ as $(x, y) \longrightarrow (x_0, y_0 \pm 0)$, where $(x_0, y_0) \in S$. Formula (3) implies that $|l_+| = |l_-| = \infty$ and $x(y)$ is not differentiable at the singular point. If $x(y)$ is extendable to (x_0, y_0) so that $(x(y_0), y_0) = (x_0, y_0)$ and if $l_- l_+ = -\infty$, then (x_0, y_0) is a local extremum of the extended curve $x(y)$. If $l_- l_+ = +\infty$, then (x_0, y_0) is not an extremum point.

1. EXTENSION OF THE CURVE $x(y)$ TO S

For the van der Pol equation, the singular set $S = \{(x, y) \in \mathbb{R}^2 : x = \varepsilon y(1 - x^2), y \neq 0, \varepsilon > 0\}$ can be represented by the curve

$$y = x/\varepsilon(1-x^2), \quad x \neq 0, \quad x \neq \pm 1, \qquad (4)$$

which splits into three branches. It follows from (3) that

$$\frac{dx}{dy} = y/(-x + \varepsilon y(1-x^2)). \qquad (5)$$

First, we consider $x(y)$ on the y axis outside S, where $\frac{dx}{dy} = 0$. Taking the second derivative at $y = 0$ yields

$$\left.\frac{d^2x(y)}{dy^2}\right|_{y=0} = \frac{-1}{x^2}, \quad x \neq 0; \tag{6}$$

i.e., on the axis $y = 0$, $x(y)$ has a local maximum if $x > 0$ and a local minimum if $x < 0$.

Following [9], we show that $x(y)$ can be extended to any singular point (x_0, y_0) by extending local first integrals $I(x, y, x_0, y_0)$. Let $x = x_0$ and $y = y_0$ be arbitrary fixed values excluding the singular point $(x_0, y_0) = (0, 0)$. Setting $\tau = t - t_0[x_0 = x(t_0), y_0 = y(t_0)]$, for system (1), we have $x = x_0 + y\tau_0 + o(\tau)$ and $y = y_0 + (-x_0 + \varepsilon y(1 - x_0^2))\tau + o(\tau)$, which yields the local first integral $I = \frac{x - x_0}{y} - \frac{y - y_0}{-x_0 + \varepsilon y(1 - x_0^2)} = C$ in $O\backslash S$, where O is a sufficiently small neighborhood of (x_0, y_0). For a fixed constant C, this neighborhood contains the unique local solution $x = x_0 + \frac{y(y - y_0)}{-x_0 + \varepsilon y(1 - x_0^2)} + yC$. Choosing $(x_0, y_0) \in S$ and extending the local solution to the singular point, we encounter a 0/0 indeterminate form. Setting $y_0 = x_0/\varepsilon(1 - x_0^2) = y^*$, we calculate

$$\lim_{u \to y^*} \frac{y(y - y^*)}{-x_0 + \varepsilon y(1 - x_0^2)} = \frac{x_0}{(\varepsilon(1 - x_0^2))^2},$$
$$x_0 \neq \pm 1.$$

Thus, the local first integral is extendable to $(x_0, y_0) \in S$ and the local curve passes through the singular point if $C = -(\varepsilon(1 - x_0^2))^{-1}$. In other words, the extended local curve satisfies

$$x(y) = x_0 + y\left[\frac{y - y_0}{-x_0 + \varepsilon y(1 - x_0^2)} - \frac{1}{\varepsilon(1 - x_0^2)}\right] \longrightarrow x_0$$

as $y \longrightarrow y_0$, $x_0 \neq \pm 1$, $x_0 \neq 0$ (otherwise, $y_0 = 0$ and we obtain a singular point). Note that (0, 0) and the lines $x = \pm 1$ lie outside S.

By using formula (5), we show that the extended curves $x(y)$ have local extrema at any singular point (x_0, y_0): the signs of $dx/dy(x_0, y_0 - 0)$ and $dx/dy(x_0, y_0 + 0)$ are opposite. Obviously, it suffices to consider the following four situations: (i) $[x_0 > 1, y_0 < 0]$; i.e., $x(y)$ has local minima; (ii) $[x_0 > 0, y_0 > 0]$; i.e., $x(y)$ also has local minima at singular points; (iii) $[x_0 < 0, y_0 < 0]$; i.e., there are local maxima; and (iv) $[x_0 < -1, y_0 > 0]$; i.e., there are also local maxima. The function $x(y)$ is not differentiable at extremal points outside $y = 0$.

2. STABILITY OF LOCAL EXTREMA

Along with Eq. (2), we consider the perturbed equation

$$y^0 = P(x, y) + \lambda\pi(x, y), \quad x^0 = y, \quad \pi \in C^1 \tag{7}$$

with the singular set

$$S^* = \{(x, y) \in \mathbb{R}^2 : P(x, y) + \lambda\pi(x, y) = 0, y \neq 0\}$$

and with integral curves $x^*(y)$ extended to S^*.

Definition 1. Let $O(x_0, y_0)$ be a neighborhood of the singular point (x_0, y_0). Curves $x(y)$ and $x^*(y)$ are said to behave similarly in $O(x_0, y_0)$ if (i) $x(y)$ increases (decreases) at some point $(x(y), y) \in O(x_0, y_0) \cap S$ and $x^*(y)$ increases (decreases) at every point $(x^*(y), y) \in O(x_0, y_0) \cap S^*$ or (ii) $x(y)$ has a local maximum (minimum) at every point $(x(y), y) \in O(x_0, y_0) \cap S$ and $x^*(y)$ has a local maximum (minimum) at every point $(x^*(y), y) \in O(x_0, y_0) \cap S^*$.

Definition 2. A point $(x_0, y_0) \in S$ is a locally stable point of $x(y)$ with respect to the transition from Eq. (2) to Eq. (7) if, for sufficiently small $|\lambda| > 0$ $(0 < \lambda < \lambda_0(x_0, y_0))$, there exists a neighborhood $O(x_0, y_0)$ of (x_0, y_0) such that the integral curves $x(y)$ and $x^*(y)$ behave similarly in $O(x_0, y_0)$. The following result was proved in [10].

Theorem. *Let $x(y)$ be an integral curve of Eq.* (2) *extended to S by using the first integral of Eq.* (2), *and let $x^*(y)$ be an integral curve of Eq.* (2) *extended to S^* by using the first integral of Eq.* (7). *Then the local extrema of $x(y)$ on S are locally stable, and the points of $x(y)$ of other types are unstable on S.*

3. SOME CONCLUSIONS

Consider the case where the plane is divided into three parts: R_1 (above the branch S_2^+ of the singular set), R_2 (below $S_2^- \subset S$), and R (between S_2^- and S_2^+). Thus, the middle branch of S between the asymptotes $x = \pm 1$ is in R. The extended curves $x(y)$ have the following properties.

1. $dx/dy < 0$ in both R_1 and R_2 (see (5)). An integral curve $x(y)$ through a point in R_1 that reaches S_2^+ cannot cross S_2^+ in the case of continuous $y(x)$, since it has a local minimum on S_2^+. Since $dx/dy < 0$ everywhere in R_1 and $(dx/dy)(x_0, y_0 + 0) \longrightarrow +\infty$ for $(x_0, y_0) \in S_2^+$, we conclude that the integral curve cannot return to R_1 after passing (x_0, y_0). Moreover, (x_0, y_0) cannot be a stop point, because the location of x_0 cannot be preserved with time if $dx/dt = y_0 \neq 0$. Assume that the model is such that a jump in $x(t)$ is impossible; i.e., $y = \dot{y}$ exists for any t. Then the integral curve is either extended from $(x_0, y_0) \in S_2^*$ by passing at $t = t_0$ into another inte-

gral curve in R, which corresponds to a different solution $x = f(t)$, or $y(t)$ has a jump $\Delta y = y_1 - y_0 \in (x_0, y_1) \in R$) at $t = t_0$ and dy/dt exists only in the class of generalized functions. This situation occurs on a singular set of the first type [6, 7]. It is easy to see that cycles entirely belonging to R_1 are impossible. A similar situation is observed for $x(y)$ that passes through a point in R_2 and reaches S_2^-. Specifically, cycles that entirely belong to R_2 are impossible. Integral curves (in particular, cycles) that intersect S_2^+ or S_2^- can have jumps in y at singular points $(x_0, y_0) \in S_2^* \cup S_2^-$.

2. In R, $dx/dy < 0$ if $x > 1$ and $y > 0$ or $x < -1$ and $y < 0$, and $dx/dy > 0$ if $-1 < x < 1$. According to the previous item, $x(y)$ through points in R can enter neither R_1 nor R_2 without a jump in $y(t)$. Various types of behavior of $x(y)$ are possible in R. Recall also that on the axis $y = 0$, the function $x(y)$ has a local maximum if $x(0) > 0$ and a local minimum if $x(0) < 0$. The intervals where this function is concave down and concave up are determined using the formula

$$\frac{d^2x}{dy^2} = \frac{x^2 + y^2 + \varepsilon xy(x^2 + 2y^2 - 1)}{[-x + \varepsilon y(1 - x^2)]^3}.$$

Figure 2 shows an example of a jump-free cycle in R.

3. If $|y| = |\dot{x}|$ takes very large but finite values, then cycles of the relaxation oscillator type arise in the plane of (x, y). For large ε, this situation has been studied by different methods in different variables, for example, in [1].

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