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Calculation of the Resonance Frequencies of an Open Dielectric Axially Symmetric Resonator of Piecewise Constant Radius

A. N. Bogolyubov, A. I. Erokhin, I. E. Mogilevskii, and N. E. Shapkina

Department of Mathematics, Faculty of Physics, Moscow State University, Moscow, 119899 Russia

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Abstract—An algorithm is suggested for calculating the resonance frequencies of open dielectric axially symmetric resonators of piecewise constant radius suitable for an arbitrary layered axially symmetric dielectric filling. The resonator is placed into a case with ideally conducting walls of a sufficiently large radius. Cases are analyzed where the resonator structure is partly coated with a perfect conductor.

Key words: modeling, resonator, frequencies, axially symmetric.

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INTRODUCTION

One of the main characteristics of a resonator is its resonance frequency spectrum, which determines its quality (Q-) factor [1]. Up-to-date technologies are capable of making resonators of almost any shape and size. It has recently become possible to produce dielectrics of any dielectric permeability. So, when designing high-Q resonance systems, there is no longer any need to search for suitable materials, and no practical problems arise in creating any dielectric resonator. Thus, the problem of searching for the optimal electrodynamic and geometrical parameters of resonance structures for attaining a maximum Q-factor at a minimum cost comes to the foreground. The need therefore arises to make a preliminary model of such systems mathematically. Some technical equipment requires the use of axially symmetric resonators with a piecewise constant radius. These can presumably be used in constructing low-loss conductor systems. In particular, such resonators are being experimentally investigated at the Chair of Oscillations of the Faculty of Physics of Moscow State University [2]. In this work, we calculate the resonance frequencies of open dielectric axially symmetrical resonators with a piecewise constant radius.

It was V.P Modenov who came up with the idea of introducing dielectric layers into a hollow resonator in order to improve its quality factor. By using mathematical modeling, Modenov and Chulkov [3] demonstrated that the Q-factor of a spherical resonator could actually be materially improved by filling it with an appropriate dielectric. They also calculated the resonance frequencies of systems with a layered axially symmetric filling.

PROBLEM STATEMENT

Consider a dielectric resonator bounded by an axially symmetric surface with a piecewise constant radius (Fig. 1). We introduce a cylindrical coordinate system (ρ, φ, z), with the origin placed at the center of the left-hand end face of the resonator and the z-axis directed along its symmetry axis. The spatial structure of the resonator can be obtained by rotating the domain Ω about the z-axis (Fig. 2). We designate the dielectric permeability by ε and assume that there is no loss in the dielectric; i.e., ε is taken to be a real quantity assuming positive values.

The spectrum of an open system is continuous [4]. Consider a mathematical model allowing one to solve the eigenvalue problem with a discrete spectrum.

We bound the space containing the resonator by a perfectly conducting cylinder whose size is substantially greater than that of the resonator; the axis of the cylinder is aligned with that of the original structure.

The natural oscillation of the closed resonator thus obtained is described by the nontrivial solution of a homogeneous boundary-value problem for a system of Maxwell's equations. The boundary conditions comprise the zero tangential component of the electric field vector on the surface of the cylinder. The conjugation boundary condition on the dielectric surface is the con-

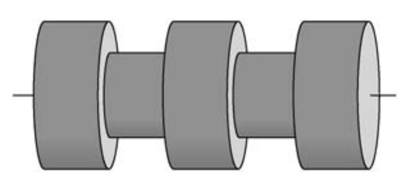

Fig. 1. Axially symmetric structure of piecewise constant radius assuming two values for five sections.

tinuity of the electric and magnetic field vector components tangent to this surface.

Consider electric oscillation. The solution components of the given system may in this case be represented in the form

$$E_z = \frac{\partial^2 U}{\partial z^2} + k^2 U, \quad H_z = 0,$$

$$E_\varphi = \frac{1}{\rho}\frac{\partial^2 U}{\partial z \partial \varphi}, \quad H_\varphi = ik\varepsilon\frac{\partial U}{\partial \rho},$$

$$E_\rho = \frac{\partial^2 U}{\partial z \partial \rho}, \quad H_\rho = -\frac{ik\varepsilon}{\rho}\frac{\partial U}{\partial \varphi},$$

where U is the Borgnis potential satisfying the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2} + k^2\varepsilon U = 0.$$

Separating the variables, we represent the solution in the form

$$U = u(\rho, z)e^{im\varphi}, \quad m = 0, \pm 1, \pm 2, \ldots$$

We will then have the following equation for u:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) - \frac{m^2}{\rho^2}u + \frac{\partial^2 u}{\partial z^2} + k^2\varepsilon u = 0,$$

which may be reduced to the form

$$Lu - \frac{m^2}{\rho}u + \rho k^2 \varepsilon u = 0, \quad (1)$$

where $Lu \equiv \operatorname{div}(\rho \operatorname{grad} u)$.

The divergence here is understood to be an operation in the two-dimensional rectangular coordinate system (ρ, z). We denote by Q the region inside the rectangle, whose rotation about the z-axis generates the cylinder, its boundary being designated ∂Q. The boundary condition will then assume the form

$$u|_{\partial Q/Oz} = 0, \quad (2)$$

and the conjugation conditions at the boundary $\partial\Omega$ will be written as

$$\varepsilon_1 \frac{\partial u_1}{\partial n}\bigg|_{S_\perp} = \varepsilon_2 \frac{\partial u_2}{\partial n}\bigg|_{S_\perp}, \qquad u_1|_{S_\parallel} = u_2|_{S_\parallel}, \quad (3)$$

where $S_\parallel$ is the part of the boundary $\partial\Omega$ that is parallel to the z-axis; $S_\perp$ is normal to it, and u_1 and u_2 are the potentials inside and outside of the dielectric, respectively.

On the part of the boundary ∂Q that lies on the z-axis and does not appear in the description of the three-dimensional problem, we set a condition of the form

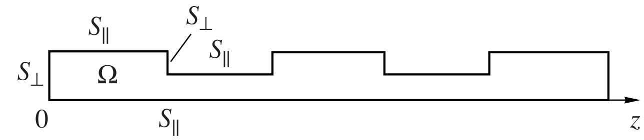

Fig. 2. Longitudinal section of an axially symmetric structure of piecewise constant radius, assuming two values for five sections.

$$\frac{\partial u}{\partial n}\bigg|_{z \cap \partial Q} = 0. \quad (4)$$

Thus, the problem of seeking the natural frequencies of a dielectric resonator bounded by a perfectly conducting cylinder is reduced to an eigenvalue problem (1)–(4) in a two-dimensional region. We multiply the latter equation by the function v and integrate it over Q. The first Green formula has the form

$$\int_Q vLud\omega = \oint_{\partial Q} \rho v \frac{\partial u}{\partial n} ds - \int_Q \rho(\operatorname{grad} u, \operatorname{grad} v) d\omega.$$

Considering the boundary condition $\frac{\partial u}{\partial n}\Big|_{z \cap \partial Q} = 0$ and assuming that

$$v|_{\partial Q/Oz} = 0,$$

we find that the surface integral is zero.

Thus, proceeding from equation (1), we arrive at the following relation for u:

$$\int_Q \rho(\operatorname{grad} u, \operatorname{grad} v) dq + m^2 \int_Q \frac{uv}{\rho} dq = k^2 \int_Q \varepsilon \rho u v dq.$$

We denote by X the space of the functions $u \in H^1$ satisfying boundary conditions (2)–(4).

We now come to the formulation of the problem on the calculation of the natural frequencies of the given resonator: we wish to find the nontrivial functions $u \in X$ and their respective k^2 values satisfying equation (3) for all functions $v \in X$.

In his monograph, Ciarlet [5] demonstrated that a solution of the problem stated does exist.

NUMERICAL SOLUTION OF THE PROBLEM

To numerically solve the problem, we will use the finite element method [6]. We denote by F_h the family of partitions of the domain Q into triangles T no more than h $(h \in H)$ in diameter. We require that the following conditions be satisfied for this family of partitions:

$$\overline{Q} = \bigcup_{T \in F_h} T, \quad \text{quad} \inf_{h \in H} h = 0, \quad \inf_{h \in H} \min_{T \in F_h} \frac{\rho_T}{h_T} > 0,$$

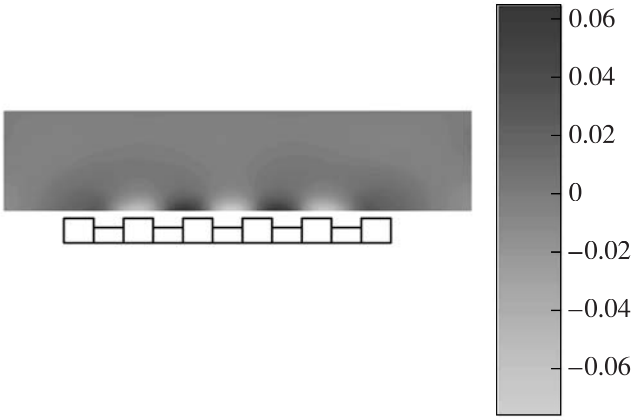

Fig. 3. Field distribution in an axially symmetric structure of piecewise constant radius, assuming two values for 11 sections at a frequency of 55 GHz.

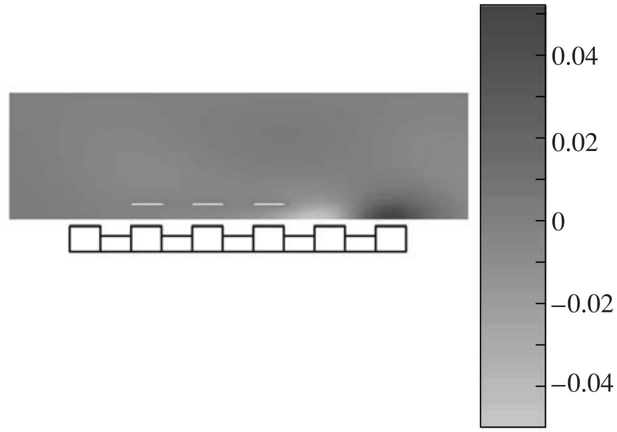

Fig. 4. Field distribution in an axially symmetric structure of piecewise constant radius, assuming two values for 11 sections, 3 of them coated with a perfect conductor (indicated by white dashes) at a frequency of 41 GHz.

where h_T is the diameter of the triangle T, ρ_T is the diameter of the circle inscribed in this triangle, $h = \max\limits_{T \in F_h} h_T$. We take the Courant pyramid [6] to serve as a finite carrier on the triangle T.

We introduce the following designations:

$$a(u, v) = \int_Q \rho(\mathrm{grad}\, u, \mathrm{grad}\, v) d\omega + m^2 \int_Q \frac{uv}{\rho} d\omega,$$

$$b(u, v) = \int_Q \varepsilon \rho u v d\omega.$$

In this case, equation (3) will assume the form $a(u, v) = k^2 b(u, v)$. It follows from the continuity of the coefficients appearing in the expressions for a and b that these are continuous bilinear forms. At the same time, the form a satisfies the condition

$$\sup_{\{v_h \in X_h : \|v_h\|_X = 1\}} a(u_h, v_h) \geq k \|u_h\|_X \ \forall u_h \in X_h,$$

where X_h is the projection of the space X on the space of the constructed finite-dimensional functions with the finite carrier. Any element v from the space X_h can be approximated however closely by the element v_h from the space X_h, provided that h is sufficiently small, i.e., $\lim\limits_{h \to 0} \|v - v_h\|_{1Q} = 0$. The solution constructed with the aid of the selected finite elements converges to the exact solution as h tends to zero [7].

RESULTS

We select from the solutions of the problem stated above the frequencies that correspond to the field concentrated mainly in the dielectric. Our numerical experiment showed them to be practically independent of the size of the conducting cylinder and its placement relative to the dielectric.

As a test, we computed the frequencies for an axially symmetric structure of a piecewise constant radius taking on a single value, i.e., for a cylinder entirely coated with a perfect conductor. An analytical solution of this test problem exists. To obtain a numerical solution, we used two programs. One is the HFSS program [8], and the other was written on the basis of the algorithm suggested above. The results obtained were compared with the exact solutions of the problem stated. It turned out that even for high-frequency modes, the error of the frequencies found was no more than one percent; the computation speed is higher with the program written on the basis of the method suggested.

Based on the mathematical modeling of a dielectric resonator, we obtained the following results:

1. A frequency spectrum is constructed for an entirely metal-coated ebonite cylinder, which agrees well with the experimental data.

2. Natural frequencies are computed for an open axially symmetric structure comprising eleven 0.9-cm-long sections of piecewise constant radius assuming two values—1 cm and 0.6 cm.

3. Frequencies are found at which electromagnetic field is practically entirely concentrated in the dielectric (Fig. 3). When a few sections of such a structure are metallized, the field can concentrate only in some sections, and not necessarily in the metal-coated ones

Fig. 5. Longitudinal section of a three-layer axially symmetric structure of piecewise constant radius, assuming two values for 11 sections.

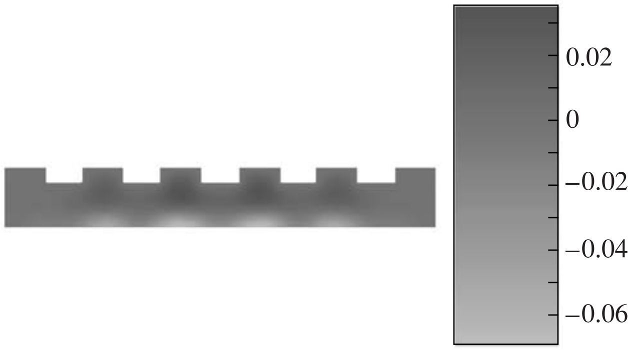

Fig. 6. Field distribution in a three-layer axially symmetrical entirely metal-coated structure of piecewise constant radius, assuming two values for 11 sections at a frequency of 136 GHz.

(Fig. 4). The excitation of individual sections was also observed in experiments with such structures.

4. A structure was analyzed whose geometry is as described above, but with two 9.9-cm-long dielectric layers 0.2 cm in radius added (Fig. 5); the entire structure is coated with a perfect conductor. The dielectric permeability of the bottom layer is $\varepsilon = 6$, that of the middle layer, $\varepsilon = 4$, and of the original structure, $\varepsilon = 2$. It is demonstrated that frequencies exist at which only a few sections get excited; the field is concentrated both in the added layer of higher dielectric permeability and in the original structure (Fig. 6).

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