ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 2, pp. 150–151. © Allerton Press, Inc., 2010.
Original Russian Text © A.N. Bogolyubov, A.A. Potapov, S.Sh. Rehviashvili, 2010, published in Vestnik Moskovskogo Universiteta. Fizika, 2010, No. 2, pp. 80–81.

A Fractional Integro–Differentiation Interpretation of the Solution of a Diffusion-Wave Equation

A. N. Bogolyubov[a], A. A. Potapov[b], and S. Sh. Rehviashvili[c]

[a] Chair of Mathematics, Faculty of Physics, Moscow State University, Moscow, 119991 Russia
e-mail: bogan7@yandex.ru
[b] Institute of Radio Engineering and Electronics of Russian Academy of Sciences, Mokhovay Moscow, 125009 Russia
e-mail: potapov@mail.cplire.ru
[c] Chair of Materials and Components of Solid-state Electronics, Faculty of Microelectronics and Computer Technologies, Kabardino-Balkar University, Chernyshevskogo, Nalchik, Kabardino-Balkar Republic, 360004 Russia
e-mail: rsergo@mail.ru

Received October 28, 2009; in final form December 23, 2009

Abstract—An example of the application of fractional integro-differentiation is presented. A method for interpreting the solution of a diffusion-wave equation was proposed in [1].

Key words: electrodynamics, fractional operators, fractional integro-differentiation.

DOI: 10.3103/S0027134910020141

The use of fractional integro–differentiation in classical thermodynamics problems has been previously shown [1]. In particular, equations with changeable types (diffusion-wave equations) have been established for vector and scalar potentials. Let us analyze the properties of a free dielectric electromagnetic field with constants ε and μ, considering a diffusion-wave equation as the base. We write the one-dimensional equation

$$\partial_{0t}^{2\alpha} u(x, t) - \frac{(c\tau)^2}{\varepsilon\mu} \frac{\partial^2 u(x, t)}{\partial x^2} = 0, \quad (1)$$

where $u(x, t)$ stands either for A, or φ, t is dimensionless time (referred to τ).

Equation (1) is linear and its particular solution can be presented as

$$u(x, t) = u_0 \exp(ikx) z(t), \quad (2)$$

where $z(t)$ is an unknown function, u_0 is a complex amplitude, and k is the component of the wave vector directed along x.

Upon inserting (2) into (1) and we obtain the equation

$$\partial_{0t}^{2\alpha} z(t) - \omega^2 z(t) = 0, \quad (3)$$

where $\omega = ck\tau/\sqrt{\varepsilon\mu}$ is the dimensionless frequency.

The particular solution of (3) is given by

$$z(t) = E_{2\alpha}(-\omega^2 t^{2\alpha}), \quad (4)$$

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\beta + 1)},$$

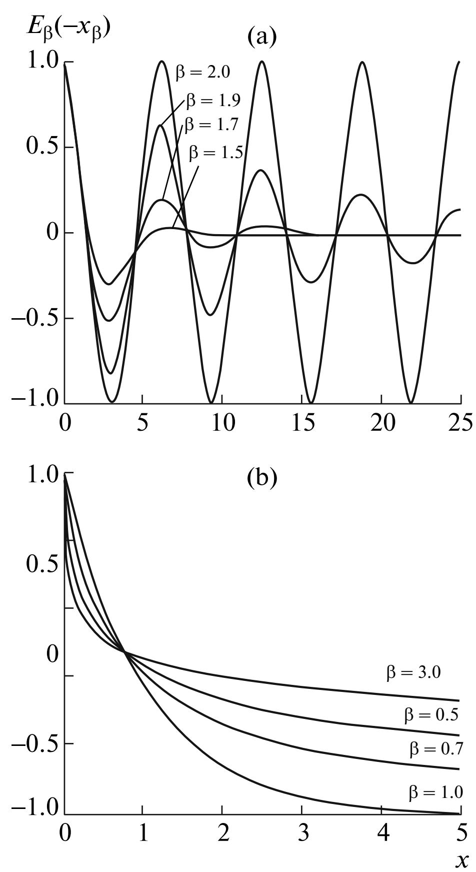

Fig. 1. Plots of the Mittag–Leffler functions for different β values.

where $E_\beta(x)$ is the Mittag–Leffler function. From (2) and (4) it follows that

$$u(x, t) = u_0 \exp(ikx) E_{2\alpha}(-\omega^2 t^{2\alpha}). \quad (5)$$

Fig. 1a and Fig. 1b present examples of $E_\beta(x)$ plots. If α from (5) lies within the interval from 1/2 to 1, then the function is periodic with respect to t and the frequency ω. If α is between 0 and 1/2, then the function monotonically decreases. The parameters α and τ specify the decrease speed.

Let us consider extreme cases of (5) in order to obtain a vivid interpretation of the equation. When $\alpha = 1$ (the hyperbolic case), using

$$E_2(x) = \cosh(\sqrt{x})$$

for the solution of (1), we write

$$u(x, t) = u_0 \exp(i(kx - \omega t)). \quad (6)$$

Expression (6) specifies a plane monochromatic wave, which is a periodic function, both of x and t.

When $\alpha = 1/2$ (the parabolic case), the following expressions are true

$$E_1(x) = \exp(x),$$
$$u(x, t) = u_0 \exp(ikx) \exp(-\omega^2 t). \quad (7)$$

The solution of (7) is periodic only with respect to x. It can be also interpreted as a plane wave with an amplitude that decreases as time passes. As this takes place, the time required for a decrease by a factor of e is $t_0 = \varepsilon\mu/(c^2 k^2 \tau)$.

Thus, in the case considered, the fractional integro–differentiation and corresponding phenomenological parameter α respond to the influence of fractional properties of charges moving in a dissipative medium upon the generation of an electromagnetic field. As α diminishes, electromagnetic waves decay. The decay asymptote $E_{2\alpha}(-t^{2\alpha}) \propto t^{-2\alpha}/\Gamma(1 - 2\alpha)$ is the power of slow diffusion wandering ($\alpha < 1/2$), which is typical for many fractional systems [2].

REFERENCES

1. A. N. Bogolyubov, A. A. Potapov, and S. Sh. Rekhviashvili, "An Approach to Introducing Fractional Integro-Differentiation in Classical Electrodynamics," Vestn. Mosk. Un-Ta. Fiz. Astron, No. 4, 9 (2009) [Mosc. Univ. Phys. Bull. **64**, 365 (2009)].
2. A. A. Potapov, *Fractals in Radio Physics and Radar: Sampling Topology* (University Lib., Moscow, 2005) [in Russian].