ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 3, pp. 164–169. © Allerton Press, Inc., 2010.
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Investigation of the Trajectories of a Magnetized Particle in the Equatorial Plane of a Magnetic Dipole

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Received January 29, 2010; in final form, March 13, 2010

Abstract—The movement of a magnetized particle in the equatorial plane of a magnetic dipole is investigated. Analysis and classification of trajectory types for such particles are made. It is shown that eight different trajectory types are possible, which depend on the particle's energy and on the orientation of its magnetic dipole moment. This permits the use of an axial magnetic field to move a magnetized particle in any point of the magnetic equatorial plane.

Key words: magnetic dipole moment, magnetic field, equation of trajectory, first integrals.

DOI: 10.3103/S0027134910030021

INTRODUCTION

Recently, interest in problems of the movement of nanoparticles in external magnetic fields has grown significantly, due to advances in nanotechnology. The reason for this is that the influence of a magnetic field on the dipole moments of particles is practically the only way to control their movement when constructing different nanostructures from magnetic materials [1, 2].

Knowledge of the basic laws of movement of massive magnetic dipoles in heterogeneous magnetic fields is vital for performing purposeful transport.

In a nonrelativistic case the equation of motion for such dipoles takes the form [3]:

$$m\frac{d^2\mathbf{r}}{dt^2} = (\mathbf{M}_1\nabla)\mathbf{H}, \qquad \frac{d\mathbf{L}}{dt} = [\mathbf{M}_1\mathbf{H}], \tag{1}$$

where m is the particle's mass, $\mathbf{M}_1$ is its magnetic dipole moment, and $\mathbf{L}$ is its total angular momentum.

These equations are appreciably nonlinear and in the general case can be solved only numerically. However, there are cases where Eqs. (1) can be solved exactly. Let us consider one of these cases.

Assume that an external magnetic field is generated by the fixed magnetic dipole $\mathbf{M}_2$:

$$\mathbf{H} = \frac{3(\mathbf{M}_2\mathbf{r})\mathbf{r} - \mathbf{M}_2 r^2}{r^5}. \tag{2}$$

Assume also that a neutral particle of mass m with a magnetic dipole moment $\mathbf{M}_1$ moves in the field. It follows from the last equation of Eqs. (1) that the movement of the particle will not be accompanied by reorientation of its magnetic dipole moment if the vector $\mathbf{M}_1$ is collinear with the external magnetic field vector $\mathbf{H}$ along the whole trajectory. This condition is fulfilled, for example, when the particle moves in the magnetic equatorial plane of the field (2), and its vector $\mathbf{M}_1$ is collinear with the vector $\mathbf{M}_2$. Let us study possible trajectory types for such movement.

1. Equations of Motion for a Particle in a Magnetic Equatorial Plane

Suppose a magnet, the mass of which is much larger than m, generates an external magnetic field (2). We place the origin of the coordinates in the center of the magnet, having directed the axis Z along the vector $\mathbf{M}_2$. Assume that a massive particle with magnetic dipole moment $\mathbf{M}_1$ moves in the external magnetic field (2) equator, and that the vector $\mathbf{M}_1$ is collinear with the vector $\mathbf{M}_2$. Then the simultaneous Eqs. (1) take the form:

$$m\frac{d^2\mathbf{r}}{dt^2} = \frac{3(\mathbf{M}_1\mathbf{M}_2)\mathbf{r}}{r^5}, \tag{3}$$

where $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$.

The simultaneous equations in (3) have two first integrals: the energy integral

$$\frac{m\dot{\mathbf{r}}^2}{2} + \frac{(\mathbf{M}_1\mathbf{M}_2)}{r^3} = E_0$$

and the angular momentum integral

$$m[\mathbf{r}, \dot{\mathbf{r}}] = m\mathbf{C}_0, \quad \mathbf{L} = \mathbf{L}_0,$$

where E_0, $\mathbf{C}_0$, and $\mathbf{L}_0$ are constants, and also

$$E_0 = \frac{m v_0^2}{2} + \frac{(\mathbf{M}_1\mathbf{M}_2)}{r_0^3}, \quad \mathbf{C}_0 = [\mathbf{r}_0\mathbf{v}_0]. \tag{4}$$

Introducing polar coordinates in the magnetic equatorial plane, we rewrite the first integrals as:

$$\frac{m}{2}\{\dot{r}^2 + r^2\dot{\varphi}^2\} + \frac{(\mathbf{M}_1\mathbf{M}_2)}{r^3} = E_0, \quad r^2\dot{\varphi} = C_0. \tag{5}$$

Proceeding to a new variable $u = 1/r$, as one does in celestial mechanics, we transform the simultaneous equations in (5) into:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2E_0}{mC_0^2} - \frac{2(\mathbf{M}_1\mathbf{M}_2)}{mC_0^2}u^3 - u^2, \qquad \frac{dt}{d\varphi} = \frac{1}{C_0u^2}. \tag{6}$$

We come thus to a task that is analogous to the problem of the movement of a massive particle in Schwarzschild space [4], but with another interaction potential.

Integrating Eqs. (6), we obtain the equation for the dipole's trajectory $\varphi = \varphi(r)$ and its equation of motion $t = t(r)$ along this trajectory:

$$\varphi(r) = \varphi_0 \pm \int_{1/r_0}^{1/r} \frac{du}{\sqrt{\Psi(u)}}, \qquad t = t_0 \pm \frac{1}{C_0}\int_{1/r_0}^{1/r} \frac{du}{u^2\sqrt{\Psi(u)}}, \tag{7}$$

where the signs of the integrals are chosen such that the initial conditions were satisfied, and

$$\Psi(u) = -\frac{2(\mathbf{M}_1\mathbf{M}_2)}{mC_0^2}u^3 - u^2 + \frac{2E_0}{mC_0^2}. \tag{8}$$

In the general case, both integrals in (7) are reduced to elliptic functions, which prevents us from visual analysis of the obtained solutions. Therefore, let us perform qualitative investigation [5] of possible trajectory types for a magnetic dipole. Since the results of such an investigation depend significantly on the sign of the scalar product ($\mathbf{M}_1$ and $\mathbf{M}_2$), we will consider both cases separately.

2. Trajectory Types for a Magnetic Dipole with Antiparallel Vectors M_1 and M_2

When vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are antiparallel, Eq. (8) takes the form:

$$\Psi(u) = \frac{2M_1M_2}{mC_0^2}u^3 - u^2 + \frac{2E_0}{mC_0^2}.$$

As is known [5], the trajectory form depends to a large extent on the number and mutual arrangement of the real roots of the equation $\Psi(u) = 0$. Since the function $\Psi(u)$ is cubic in the variable u, the equation $\Psi(u) = 0$ has three roots. According to Viet's theorem, these roots satisfy the following conditions:

$$u_1 + u_2 + u_3 = \frac{mC_0^2}{2M_1M_2}, \qquad u_1u_2 + u_1u_3 + u_2u_3 = 0, \qquad u_1u_2u_3 = -\frac{E_0}{M_1M_2}. \tag{9}$$

Calculate the discriminant D of the equation $\Psi(u) = 0$:

$$D = \frac{E_0}{(2M_1M_2)^2}\left[E_0 - \frac{m^3C_0^6}{54(M_1M_2)^2}\right]. \tag{10}$$

Since in the considered case the energy of the dipole (4)

$$E_0 = \frac{m v_0^2}{2} - \frac{M_1M_2}{r_0^3}$$

is a function of the difference between two positive values $m v_0^2/2 > 0$ and $M_1M_2/r_0^3 > 0$ with an undefined sign, the discriminant (10) is positive when $E_0 < 0$ or $E_0 > m^3C_0^6/(54M_1^2M_2^2)$; it is zero, when $E_0 = m^3C_0^6/(54M_1^2M_2^2)$; or $E_0 = 0$ and it is negative when $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$. This means that five trajectory types for a magnetic dipole are possible when the vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are antiparallel. We consider them in order.

(1) When $E_0 > m^3C_0^6/(54M_1^2M_2^2)$, the discriminant D is positive, so the cubic equation $\Psi(u) = 0$ has one real root u_1 and two conjugate complex roots $u_2 = b_1 + ic_1$ and $u_3 = b_1 - ic_1$ [6]. According to (9) the real root should be negative: $u_1 = -a_1^2 < 0$. Therefore, the function $\Psi(u)$ can be written in the present case as:

$$\Psi(u) = \frac{2M_1M_2}{mC_0^2}(u + a_1^2)[(u - b_1)^2 + c_1^2]. \tag{11}$$

It follows from (11) that with nonnegative u values the function $\Psi(u) > 0$. This is why u will be a monotonic function of the polar angle φ; this means that there will be no pivot and limit points on the trajectory. Thus, with $E_0 > m^3C_0^6/(54M_1^2M_2^2)$, the trajectory is either infinite ($u \longrightarrow 0$, $r \longrightarrow \infty$) or finite ($u \longrightarrow \infty$, $r \longrightarrow 0$), depending on the initial conditions. The actual trajectory form depends on the sign of $du/d\varphi$ at the initial time $t = t_0$. Since at $u \geq 0$ the condition $\Psi(u) \neq 0$ is fulfilled, two cases are possible.

a) $du/d\varphi > 0$ at $t = t_0$.

Then (6) takes the form:

$$\frac{du}{d\varphi} = \sqrt{\frac{2M_1M_2}{mC_0^2}(u + a_1^2)[(u - b_1)^2 + c_1^2]},$$

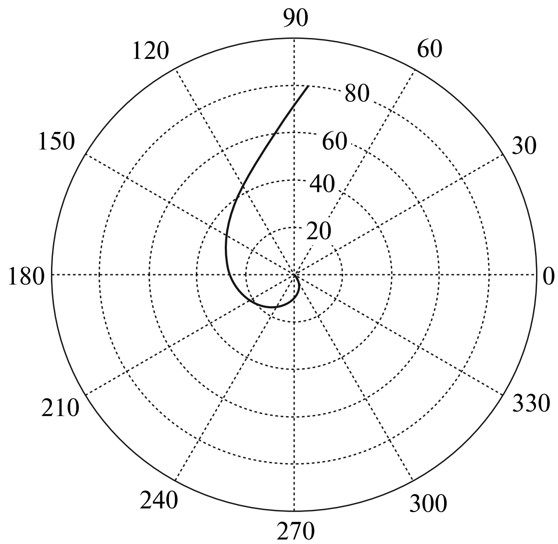

Fig. 1. A dipole's trajectory at $E_0 > m^3C_0^6/(54M_1^2M_2^2)$.

which displays the monotonous increase in u with increase in the polar angle φ.

Let us find the asymptotic expression for this trajectory at $u \longrightarrow \infty$ ($r \longrightarrow 0$, i.e., finite movement).

At $u \longrightarrow \infty$ Eq. (6) takes the form:

$$\frac{du}{d\varphi} = \sqrt{\frac{2M_1M_2}{mC_0^2}u^3}.$$

Integrating this, we find the expression for the finite part of the trajectory:

$$\frac{1}{u} = r = \frac{M_1M_2}{2mC_0^2}(\varphi - \varphi_1)^2, \qquad (12)$$

where φ_1 is the integration constant that is defined by the initial conditions; expression (12) is valid for $\varphi \le \varphi_1$.

It follows from (12) that the distance between the dipoles shortens asymptotically as the square of the difference $(\varphi - \varphi_1)$, and at polar angle $\varphi = \varphi_1$ it becomes zero.

b) $du/d\varphi < 0$ at $t = t_0$.

In this case Eq. (6) takes the form:

$$\frac{du}{d\varphi} = -\sqrt{\frac{2M_1M_2}{mC_0^2}(u + a_1^2)[(u - b_1)^2 + c_1^2]}.$$

This shows that u is a monotonously decreasing function of the polar angle φ.

At $u \longrightarrow 0$ (or $r \longrightarrow \infty$) we obtain:

$$\frac{du}{d\varphi} = -\sqrt{\frac{2M_1M_2}{mC_0^2}a_1^1[b_1^2 + c_1^2]}.$$

From this we obtain the asymptotic expression for the infinite part of the trajectory:

$$\frac{1}{u} = r = \frac{\sqrt{mC_0^2}}{(\varphi_1^* - \varphi)\sqrt{2M_1M_2a_1^2[b_1^2 + c_1^2]}},$$

where φ_1^* is the integration constant that is defined by the initial conditions.

The distance between the dipoles M_1 and M_2 grows and is inversely proportional to the square of the difference $(\varphi - \varphi_1^*)$ on the infinite part of the trajectory, and at a finite polar angle $\varphi = \varphi_1^*$ it becomes infinite.

A typical view of the dipole's trajectory at $E_0 > m^3C_0^6/(54M_1^2M_2^2)$ is given in Fig. 1.

(2) At $E_0 = m^3C_0^6/(54M_1^2M_2^2)$ the discriminant $D = 0$ and the equation $\Psi(u) = 0$ has three real roots, two of which coincide with each other. In this case (6) takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2M_1M_2}{mC_0^2}\left(u + \frac{mC_0^2}{6M_1M_2}\right)\left(u - \frac{mC_0^2}{3M_1M_2}\right)^2. \qquad (13)$$

It is evident from this equation that the trajectory of the dipole $\mathbf{M}_1$ is predetermined by the initial conditions for u at $t = t_0$.

If at zero time $u = mC_0^2/(3M_1M_2)$, then $du/d\varphi = 0$, and the dipole will move on a circular orbit of radius $R = r_0 = 3M_1M_2/(mC_0^2)$ with frequency $\omega = m^2C_0^5/(9M_1^2M_2^2)$.

If at zero time $t = t_0$ the magnetic dipole is inside a circle of radius $R = r_0 = 3M_1M_2/(mC_0^2)$ then Eq. (13) has the following solution:

$$\frac{1}{r} = u = -\frac{mC_0^2}{6M_1M_2}\left[1 - 3\coth^2\left(\frac{\varphi - \varphi_2}{2}\right)\right], \qquad (14)$$

where φ_2 is the integration constant that is defined by the initial conditions.

It follows from the above expression that $u \longrightarrow \infty$ (or $r \longrightarrow 0$) at $\varphi = \varphi_2$; and at $\varphi \longrightarrow \pm\infty$ the dipole will asymptotically draw near to the circle of radius $R = r_0 = 3M_1M_2/(mC_0^2)$, making an infinite number of revolutions.

If at zero time $t = t_0$ the magnetic dipole is outside the circle of radius $R = r_0 = 3M_1M_2/(mC_0^2)$ then Eq. (13) has the following solution:

$$\frac{1}{r} = u = \frac{mC_0^2}{6M_1M_2}\left[3\tanh^2\left(\frac{\varphi - \varphi_2^*}{2}\right) - 1\right], \qquad (15)$$

where φ_2^* is the integration constant that is defined by the initial conditions.

Since the variable u must be nonnegative, it follows from this expression that at polar angle φ tending the value $\varphi = \varphi_k$, which satisfies the condition $\tanh^2((\varphi_k - \varphi_2^*)/2) = 1/3$, the magnetic dipole recedes to infinity; at $\varphi - \varphi_2^* \longrightarrow \pm\infty$ it asymptotically draws near to the

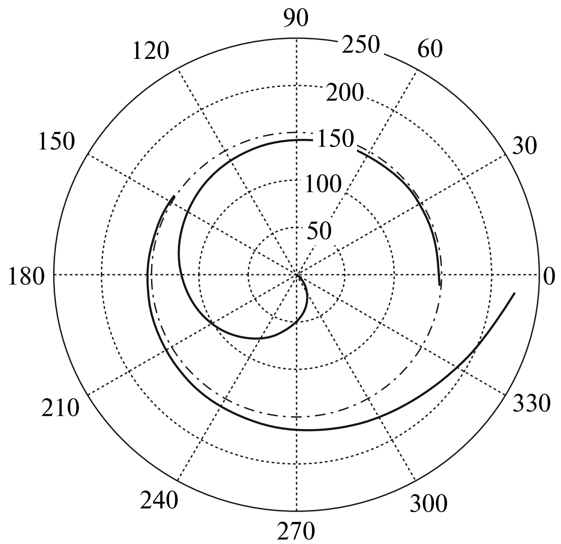

Fig. 2. Examples of possible trajectories that are described by Eqs. (14) and (15).

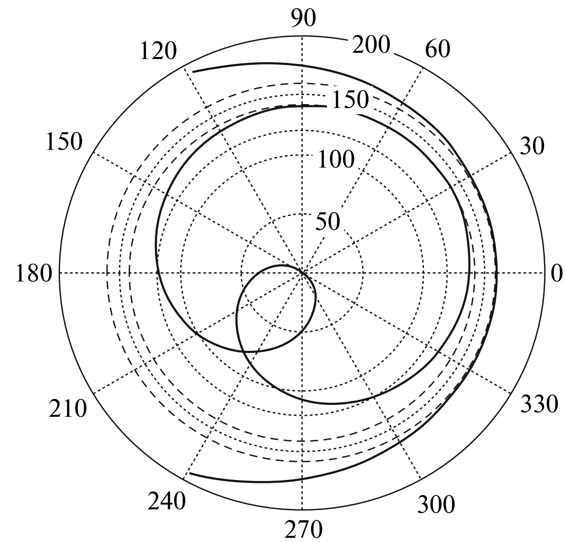

Fig. 3. An example of a trajectory at $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$.

circle of radius $R = r_0 = 3M_1M_2/(mC_0^2)$, making an infinite number of revolutions.

Examples of the possible trajectories that are described by Eqs. (14) and (15) are shown in Fig. 2.

(3) At $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$ the discriminant D is negative and the equation $\Psi(u) = 0$ has three different real roots. According to the last two expressions in (9) one of the roots must be negative and the other two must be positive: $u_1 = a_3^2 > 0$, $u_2 = b_3^2 > 0$, and $u_3 = -c_3^2 < 0$. Then Eq, (6) takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2M_1M_2}{mC_0^2}(u - a_3^2)(u - b_3^2)(u + c_3^2). \quad (16)$$

Without loss of generality we can assume that $b_3^2 < a_3^2$. Therefore, the requirement $\Psi(u) \geq 0$ will determine the following regions of a dipole's movements: $0 \leq u \leq b_3^2$ and $a_3^2 \leq u \leq \infty$. In the region $b_3^2 \leq u \leq a_3^2$ movements with energy $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$ are impossible, because $\Psi(u) < 0$ there.

It follows from Eq. (16) that depending on the initial conditions, the dipole can execute either a finite motion inside a circle of radius $R_1 = 1/a_3^2$ or an infinite motion outside a circle of radius $R_2 = 1/b_3^2$. Examples of these trajectory types at $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$ are shown in Fig. 3.

(4) At $E_0 = 0$ the discriminant $D = 0$ and (6) takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \left[\frac{2M_1M_2}{mC_0^2}u - 1\right]u^2.$$

It is simple to solve this equation:

$$r = \frac{1}{u} = \frac{M_1M_2}{mC_0^2}[1 + \cos(\varphi - \varphi_4)], \quad (17)$$

where φ_4 is the integration constant that is defined by the initial conditions.

The curve that is described by Eq. (17) is called Pascal's limacon in scientific literature. A typical view of this curve is plotted in Fig. 4.

(5) At $E_0 < 0$, the discriminant D is positive and this is why the cubic equation $\Psi(u) = 0$ has one positive

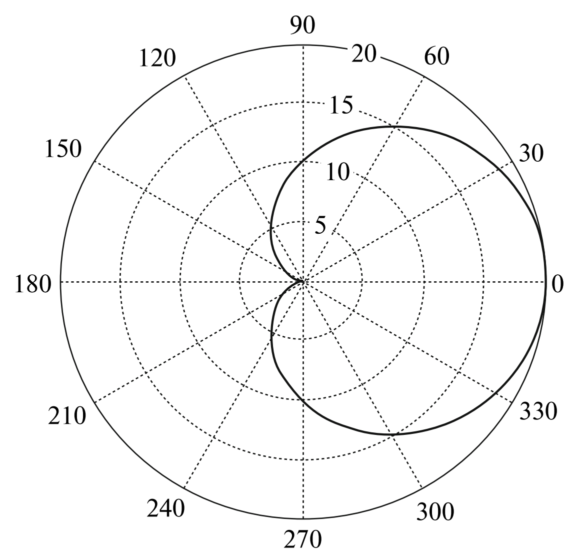

Fig. 4. Pascal's limacon.

root u_1 and two conjugate complex roots $u_2 = b_5 + ic_5$ and $u_3 = b_5 - ic_5$. According to the last expression in (9), the real root must be positive: $u_1 = a_5^2 > 0$. Therefore, in the considered case Eq. (6) takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2M_1M_2}{mC_0^2}(u-a_5^2)[(u-b_5)^2+c_5^2].$$

It follows from the condition $\Psi(u) \geq 0$ that at $E_0 < 0$ the dipole's movements will lie inside a circle of radius $R = 1/a_5^2$, and $r \leq R$. In this region, depending on the initial conditions, the dipole will either immediately move to the origin of the coordinates or it will first draw near to the circle of radius $R = 1/a_5^2$ and then reflect from it and start to move to the origin of the coordinates (Fig. 5).

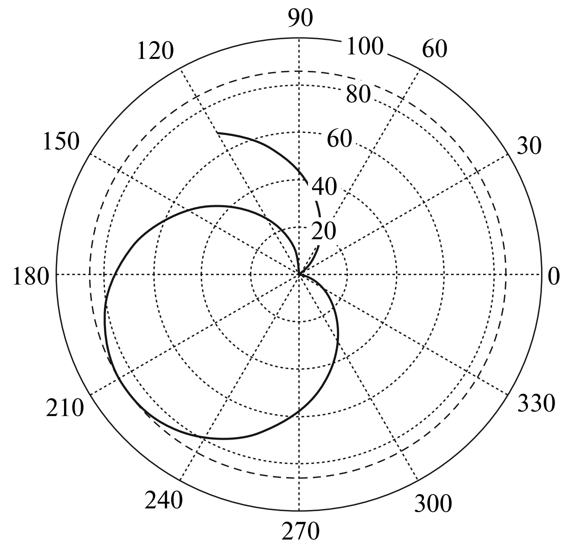

Fig. 5. The typical trajectory of a dipole at $E_0 < 0$.

3. Trajectory Types for a Magnetic Dipole with Parallel Vectors $\mathbf{M}_1$ *and* $\mathbf{M}_2$

When the vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are parallel (8) will take the form:

$$\Psi(u) = -\frac{2M_1M_2}{mC_0^2}u^3 - u^2 + \frac{2E_0}{mC_0^2}.$$

In this case Viet's Theorem yields correlations between the roots of the equation $\Psi(u) = 0$ that differ from (9):

$$u_1 + u_2 + u_3 = -\frac{mC_0^2}{2M_1M_2},$$
$$u_1u_2 + u_1u_3 + u_2u_3 = 0, \quad (18)$$
$$u_1u_2u_3 = \frac{E_0}{M_1M_2}.$$

Furthermore, it follows from (4) that with parallel vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ the energy (4) E_0 is positive: $E_0 > 0$. Therefore, the following three trajectory types are possible, which correspond to the three permitted ranges of energy E_0.

(1) At $E_0 > m^3C_0^6/(54M_1^2M_2^2)$, the discriminant D is positive and the equation $\Psi(u) = 0$ has one real and two conjugate complex roots. The last expression in (18) requires that the real root is positive. Since the function $\Psi(u)$ must be nonnegative in the region of the dipole's movements, the expression (8) takes the form:

$$\Psi(u) = \frac{2M_1M_2}{mC_0^2}(a_6^2 - u)[(u-b_6)^2 + c_6^2].$$

Consequently, the region of the dipole's movements is bounded below by the circle of radius $R = 1/a_6^2$, and $r \geq R$.

In this region, depending on the initial conditions, the dipole will either immediately move to infinity or it will first draw near to the circle of radius $R = 1/a_6^2$ and then touch it and start to move to infinity.

(2) At $E_0 = m^3C_0^6/(54M_1^2M_2^2)$, the discriminant $D = 0$, and the equation $\Psi(u) = 0$ has three real roots, two of which are negative and equal to each other, and the third is positive. In this case (8) may be rewritten as:

$$\Psi(u) = \frac{2M_1M_2}{mC_0^2}\left(\frac{mC_0^2}{6M_1M_2} - u\right)\left(u + \frac{mC_0^2}{3M_1M_2}\right)^2.$$

Thus, it is simple to find the trajectory equation from (7):

$$\frac{1}{r} = u = \frac{mC_0^2}{6M_1M_2}\left[1 - 3\tanh^2\left(\frac{\varphi - \varphi_7}{2}\right)\right],$$

where φ_7 is an integration constant that is defined by the initial conditions.

This expression shows that the magnetic dipole's trajectory starts at infinity at the angle φ value, which satisfies the condition $\tanh((\varphi - \varphi_7)/2) = -1/\sqrt{3}$; then, as the polar angle φ increases, it draws near to the circle of radius $R = 6M_1M_2/(mC_0^2)$ and touches it when $\varphi = \varphi_7$ and then it recedes to infinity at $\tanh((\varphi - \varphi_7)/2) = 1/\sqrt{3}$. The dipole's trajectory in this case looks similar to the trajectory in Fig. 6, differing from it by its curvature.

(3) At $0 < E_0 < m^3C_0^6/(54M_1^2M_2^2)$, the discriminant D is negative, and the equation $\Psi(u) = 0$ has three different real roots. It follows from the last two expressions in (18) that one of the roots must be positive, and the other two must be negative: $u_1 = a_8^2 > 0$, $u_2 = -b_8^2 < 0$,

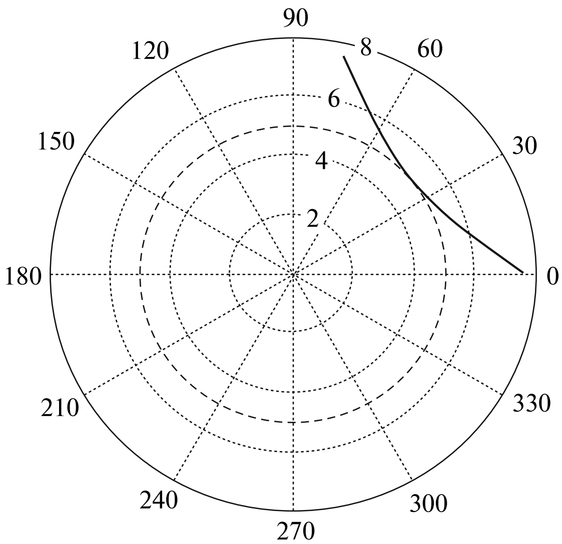

Fig. 6. The typical trajectory of a dipole at $E_0 > m^3 C_0^6/(54 M_1^2 M_2^2)$.

$u_3 = -c_8^2 < 0$. Therefore, in the considered range of energy E_0 the function $\Psi(u)$ may be rewritten as:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2M_1M_2}{mC_0^2}(a_8^2 - u)(u + b_8^2)(u + c_8^2). \qquad (19)$$

The condition $\Psi(u) \geq 0$ determines the following region of the dipole's movements: $0 \leq u \leq a_8^2$. In the region $a_8^2 \leq u \leq \infty$ movement with energy $0 < E_0 < m^3 C_0^6/(54 M_1^2 M_2^2)$ is impossible, because $\Psi(u) < 0$ there.

Equation (19) requires that the dipole executes only finite motion in the region outside the circle of radius $R = 1/a_8^2$. The trajectory of the magnetic dipole at $0 < E_0 < m^3 C_0^6/(54 M_1^2 M_2^2)$ is similar to that in Fig. 6, differing from it by its curvature.

CONCLUSIONS

Analysis shows that, depending on the initial conditions, and, first of all, on the energy E_0 of a magnetized particle, it may move on any one of eight trajectory types. Therefore it is possible to transport a nanoparticle by means of an external dipolar magnetic field to any point of the magnetic equatorial plane, thus ensuring the fulfillment of the corresponding initial conditions.

ACKNOWLEDGEMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 09-02-13540-ofi-ts.

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