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THEORETICAL AND MATHEMATICAL PHYSICS

The Phase Structure of a Generalized Gross–Neveu Model in (2+1)-Dimensional Space–Time

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Abstract—The phase structure of a (2+1)-dimensional Gross–Neveu model with four different channels of fermion–antifermion interaction, and, correspondingly, four different coupling constants, is studied. It is shown that the model describes five different phases of the interaction of planar fermions in which either spatial parity or chiral symmetry can be broken. The existence of a phase that is characterized by simultaneous spontaneous breaking of both these symmetries that was not observed earlier in such models is demonstrated.

Keywords: Gross–Neveu model, phase transitions.

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INTRODUCTION

The investigation of (2+1)-dimensional models of the quantum field theory (QFT) with four-fermion Gross-Neveu-type interaction has recently attracted great attention [1]. This interest is partly determined by the simple structure of QFT in two spatial dimensions. It is much easier to study such physical effects of the real (3+1)-dimensional world as dynamic symmetry breaking [2–4], color superconductivity [5], and other phenomena of quantum chromodynamics [3, 4] in low-dimensional QFT models due to this simplicity. Another example of this kind is the effect of spontaneous chiral symmetry breaking induced by an external (chromo)magnetic field. This phenomenon was first described on the basis of (2+1)-dimensional Gross–Neveu models [4–10]. These models are also useful for developing new tools for QFT studies. Here, the optimal perturbation theory should be mentioned to prove the above statement; this method was developed well in the framework of (2+1)-dimensional Gross–Neveu models [11, 12].

There exists, however, a more physical motivation for QFT investigation in two spatial dimensions. It is based on the observation of the fact that many systems in a condensed state have a planar crystalline structure. Moreover, it is also important that their low-energy excitation spectrum is efficiently described based on the relativistic Dirac equation, rather than the Schrödinger equation. Such systems are, for example, cuprate-based high-temperature ceramic semiconductors and graphene, a planar hexagonal crystalline lattice with a thickness of one atom consisting of carbon atoms. Since no consistent theory of high-temperature superconductivity exists, effective (2+1)-dimensional QFTs, including Gross–Neveu models [13–20], play an important role in the description of various phenomena in such planar systems, including graphene. The latter models are especially useful in the description of phase transitions in planar systems under the action of different external factors [21–25].

A gap in the energy spectrum of quasiparticles is known to be formed in the course of phase transitions; as a rule, in this case the model symmetry is spontaneously broken. In other words, single-particle excitations of the system's ground state spontaneously acquire a mass in the course of phase transition, and the form of this mass, i.e., the form of the mass term in the system's Lagrangian influences greatly the dynamics of physical processes. In the papers that were mentioned above (see, e.g., [3, 6, 23]) the structure of Gross–Neveu models, as a rule, admitted one phase transition with spontaneous breaking of chiral invariance during which a gap (mass) of one simplest form appeared in the fermion spectrum.

In this study, we investigate the phase structure of the generalized (2+1)-dimensional Gross–Neveu model with four-fermion interaction of four different types. We demonstrate that four different phase transitions are possible in this model, and either P parity or chiral invariance, or both symmetries are broken spontaneously in each of these transitions. Therefore, in the framework of this generalized model, quasiparticles may spontaneously acquire four different mass terms, i.e., a great variety of physical phenomena which can be examined in the framework of one microscopic theory are predicted.

THE MODEL AND ITS EFFECTIVE POTENTIAL

We study the (2+1)-dimensional Gross–Neveu model with the Lagrangian of the following form:

$$L = \bar{\Psi} i\hat{\partial}\Psi + \frac{G_1}{2N}(\bar{\Psi}\Psi)^2 + \frac{G_2}{2N}(\bar{\Psi}\tau\Psi)^2 + \frac{H_1}{2N}(\bar{\Psi}i\Gamma^5\Psi)^2 \frac{H_2}{2N}(\bar{\Psi}i\Gamma^3\Psi)^2, \quad (1)$$

where $\Psi(x)$ is the fundamental multiplet of the $U(N)$ group, each component of this multiplet is a four-component Dirac spinor, i.e., $\Psi^t(x) \equiv (\Psi_1^t(x),\ldots,\Psi_N^t(x))$, where t denotes transposition, and for each particular $a = 1,..,N$ $\Psi_a(x)$ is the spinor that is transformed according to a reducible four-dimensional representation of SO(2,1)[1]. (Hereinafer, the spinor indices and $U(N)$ group indices are not given explicitly, while summing over them is implied everywhere.) In models of the (1) type the coupling constants G_i, H_i are not real constants of fermion quasiparticle interaction. These parameters rather describe Coulomb interaction between quasiparticles and between quasiparticles and the crystalline lattice of the planar system.

The following notation is used in (1): $\hat{\partial} = \sum_{\mu=0}^{2}\Gamma^\mu\partial_\nu$, where $\Gamma^\mu = diag(\gamma^\mu - \gamma^\mu)$,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

The following representation is used for the 4×4 matrices Γ^3 and Γ^5:

$$\Gamma^3 = \begin{pmatrix} 0\ , & I \\ I\ , & 0 \end{pmatrix}, \quad \gamma^5 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3 = i\begin{pmatrix} 0\ , & -I \\ I\ , & 0 \end{pmatrix}, \quad (3)$$

where I is the unitary 2×2 matrix. Here, $\tau = -i\Gamma^3\Gamma^5 = diag(1,1,-1,-1)$. All Γ^μ matrices $(\mu = 0,1,2,3,5)$ anticommutate with each other and act in the four-dimensional spinor space. It should be noted that Lagrangian (1) is invariant with respect to two discrete chiral transformations, Γ^3 and Γ^5, and the space inversion operation P $(a = 1,\ldots,N)$:

$$\Gamma^3 : \ \Psi_a \to \Gamma^3\Psi_a; \ \Gamma^5 : \ \Psi_a \to \Gamma^5\Psi_a; \ P : \ \Psi_a(t,x,y) \to \Gamma^5\Gamma^1\Psi_a(t,-x,y). \quad (4)$$

It is known that Gross–Neveu models of the (1) type are renormalizable in the framework of the nonperturbative method of a $1/N$ expansion [3]. As a consequence, in this study we use this method, and it is assumed that $N \gg 1$. In this case, it is sufficient to consider the properties of theory (1) in the leading order with respect to $1/N$ only, since higher order corrections to any dynamic quantity of the model are negligible.

Below, we demonstrate that for each fixed set of coupling constants in model (1) one of the five possible phases, A, B, C, D, E, occurs. In phase A fermions are massless, while phases B, C, D, E correspond to the effective Lagrangian with a spontaneous fermion mass term of the form $\bar{\Psi}\Psi$, $\bar{\Psi}\tau\Psi$, $\bar{\Psi}i\Gamma^5\Psi$, and $\bar{\Psi}i\Gamma^3\Psi$, respectively. Since each of these mass terms breaks one or two discrete symmetries (4) of model (1), these symmetries of the model are broken spontaneously in the corresponding phases.

It is convenient to study the phase structure of model (1) using the equivalent Lagrangian with the auxiliary boson fields σ_1, σ_2, φ_1, and φ_2:

$$L[\bar{\Psi},\Psi,\sigma_i,\varphi_i] = \bar{\Psi}[i\hat{\partial} - \sigma_1 - \sigma_2\tau - i\varphi_1\Gamma^5 - i\varphi_2\Gamma^3]\Psi - N\sum_{k=1}^{2}\left(\frac{\sigma_k^2}{4G_k} + \frac{\varphi_k^2}{4H_k}\right). \quad (5)$$

The equations of motion for the auxiliary fields can be easily obtained from (5),

$$\sigma_1 = -2\frac{G_1}{N}\bar{\Psi}\Psi, \quad \sigma_2 = -2\frac{G_2}{N}\bar{\Psi}\tau\Psi, \quad \varphi_1 = -2\frac{H_1}{N}\bar{\Psi}i\Gamma^5\Psi, \quad \varphi_2 = -2\frac{H_2}{N}\bar{\Psi}i\Gamma^3\Psi. \quad (6)$$

Substituting expressions (6) into (5), we obtain initial Lagrangian (1), which, at least, means that models (1) and (5) are equivalent for the field equations of motion. This means that the properties of the vacuum state of model (1), i.e., its phase structure, can be considered in the framework of auxiliary equivalent Lagrangian (5) (this is what will be done, because it is convenient). In this case the effective action of the fields σ_i, φ_i in the leading order of $1/N$ expansion [3] has the form

$$S_{\rm eff}(\sigma_i,\phi_i) = -\int dx^0 d^2x \sum_{k=1}^{2}\left(\frac{\sigma_k^2}{4G_k} + \frac{\phi_k^2}{4H_k}\right) - i\mathrm{Tr}_{sx}\ln(i\hat{\partial} - \sigma_1 - \sigma_2\tau - i\varphi_1\Gamma^5 - i\varphi_2\Gamma^3), \quad (7)$$

[1] In the most general form, i.e., with four nonzero coupling constants G_i, H_i, model (1) was first considered in [26] (particular cases of this model with some constants equal to zero were studied in [2, 3, 21]). In [26], however, certain mistakes were made in investigation of the phase structure of the model. Since Gross–Neveu models with several different four-fermion structures are still topical for examination of, for example, graphene [31], in this study we make up for the existing lack of knowledge on the phase structure of model (1).

where the operation Tr_{sx} means the spur with respect to both the four-dimensional spinor (s) and (2+1)-dimensional coordinate (x) spaces. The structure of a vacuum is known to be determined by vacuum averages (condensates) of the boson fields σ_i and φ_i (i=1, 2) (hereinafter, the condensates of these fields are denoted as $\langle\sigma_{1,2}\rangle$ and $\langle\varphi_{1,2}\rangle$). In order to find $\langle\sigma_{1,2}\rangle$ and $\langle\varphi_{1,2}\rangle$), it is necessary to find the effective potential of the model connected with (7) using the following relation:

$$V_{\rm eff}(\sigma_i,\varphi_i)\int dx^0 d^2x = -S_{\rm eff}(\sigma_i,\phi_i)|_{\sigma_i,\phi_i=\rm const}. \quad (8)$$

According to the method that was described in [27], the following can be obtained from (7) and (8):

$$V_{\rm eff}(\sigma_i,\varphi_i) = \sum_{k=1}^{2}\left(\frac{\sigma_k^2}{4G_k}+\frac{\varphi_k^2}{4H_k}\right) + i\int\frac{dp_0 d^2\vec{p}}{(2\pi)^3}\mathrm{Tr}_s \ln D(p), \quad (9)$$

where $D(p) = p_0\Gamma^0 - \vec{p}\vec{\Gamma} - \sigma_1 - \sigma_2\tau - i\varphi_1\Gamma^5 - i\varphi_2\Gamma^3$ is the Dirac operator that is written in square brackets in formula (5) in the momentum representation. Since $\mathrm{Tr}_s \ln D(p) = \ln \mathrm{Det} D(p) = \sum_i \ln\varepsilon_i$, where ε_i are the four eigenvalues of the 4×4 matrix $D(p)$,

$$\varepsilon_{1,2,3,4} = \sigma_1 \pm\sqrt{\left(\sigma_2 \pm\sqrt{p_0^2-\vec{p}^2}\right)^2 - \varphi_1^2 - \varphi_2^2}, \quad (10)$$

we have from (9)

$$V_{\rm eff}(\sigma_i,\varphi_i) = \sum_{k=1}^{2}\left\{\frac{\sigma_k^2}{4G_k}+\frac{\varphi_k^2}{4H_k} + i\int\frac{dp_0 d^2\vec{p}}{(2\pi)^3}\ln(p_0^2-\vec{p}^2-M_k^2)\right\}, \quad (11)$$

where $M_{1,2} = \sigma_2 \pm\sqrt{\sigma_1^2+\varphi_1^2+\varphi_2^2}$. Integration over p_0 in (28) can be performed using the general relation $\int dp_0 \ln(p_0 - A) = i\pi|A|$, which is valid to an infinite term independent of the real quantity A. Therefore,

$$V_{\rm eff}(\sigma_i,\varphi_i) = \sum_{k=1}^{2}\left\{\frac{\sigma_k^2}{4G_k}+\frac{\varphi_k^2}{4H_k} - \int\frac{d^2\vec{p}}{(2\pi)^2}\sqrt{\vec{p}^2+M_k^2}\right\}. \quad (12)$$

It can be seen from (12) that $V_{\rm eff}(\sigma_i,\varphi_i)$ is an ultraviolet-diverging quantity that is first regularized by limiting the domain of integration in (12) by the condition $|\vec{p}| < \Lambda$. As a result, we obtain, to an additive constant independent of the fields,

$$V_{\rm eff}^{reg}(\sigma_i,\varphi_i) = \sum_{k=1}^{2}\left\{\frac{\sigma_k^2}{4}\left(\frac{1}{G_k}-\frac{2\Lambda}{\pi}\right)+\frac{\varphi_k^2}{4}\left(\frac{1}{H_k}-\frac{2\Lambda}{\pi}\right) + \frac{|M_k|^3}{6\pi} + |M_k|^3\, O\left(\frac{|M_k|}{\Lambda}\right)\right\}. \quad (13)$$

In order to obtain the finite $V_{\rm eff}(\sigma_i,\varphi_i)$ from regularized expression (13) in the limit $\Lambda\to\infty$, it should be additionally required that the bare coupling constants G_k, H_k depend in the following way on the cutting parameter Λ:

$$\frac{1}{G_k} = \frac{2\Lambda}{\pi} + g_k, \quad \frac{1}{H_k} = \frac{2\Lambda}{\pi} + h_k. \quad (14)$$

In this case, in the limit $\Lambda\to\infty$, we obtain from (13) the finite (renormalized) expression for the effective potential,

$$V(\sigma_i,\varphi_i) = \sum_{k=1}^{2}\left[\frac{g_k}{4}\sigma_k^2 + \frac{h_k}{4}\varphi_k^2 + \frac{|M_k|^3}{6\pi}\right]. \quad (15)$$

The coordinates of the point of global minimum of effective potential (15) determine the condensates $\langle\sigma_{1,2}\rangle$ and $\langle\varphi_{1,2}\rangle$, which play a fundamental role in investigation of the properties of the vacuum state of the system, i.e., all that is called the phase structure (symmetry of the ground sate, masses of its single-particle excitations, and so on). It can be easily seen that in our case the fermion mass terms in Lagrangian (5) appear spontaneously if the condensates $\langle\sigma_{1,2}\rangle$ and $\langle\varphi_{1,2}\rangle$ are nonzero. For this purpose, it is sufficient to perform the field shift in (5),

$$\sigma_{1,2}\to\sigma_{1,2}+\langle\sigma_{1,2}\rangle, \quad \varphi_{1,2}\to\varphi_{1,2}+\langle\varphi_{1,2}\rangle. \quad (16)$$

THE PHASE STRUCTURE OF THE MODEL

Let us examine whether the effective potential (15) has a global minimum. First of all, we note that the function $V(\sigma_i,\varphi_i)$ is even with respect to each of the four independent variables $\sigma_{1,2}$ and $\varphi_{1,2}$. Therefore, it is sufficient to study it on the set $\varphi_{1,2}\ge0$ and $\sigma_{1,2}\ge0$ (where, obviously, $M_1 \equiv \sigma_2 + \sqrt{\sigma_1^2+\varphi_1^2+\varphi_2^2} \ge 0$). Then, the structure of function (15) is such that it is convenient to search for the point of the global minimum first using the variables M_1, M_2, $x=\varphi_1^2$ and $y=\varphi_2^2$, and in the end return to the variables $\varphi_{1,2}\ge0$ and $\sigma_{1,2}\ge0$. Since $\sigma_1^2 = -x - y + (M_1-M_2)^2/4$ and

$\sigma_2^2 = (M_1 + M_2)^2/4$, in terms of the new variables effective potential (15) takes the form

$$V(M_1, M_2, x, y) = \frac{g_1}{16}(M_1 - M_2)^2 + \frac{g_2}{16}(M_1 + M_2)^2 + \frac{h_1 - g_1}{4}x + \frac{h_2 - g_1}{4}y + \frac{|M_1|^3}{6\pi} + \frac{|M_2|^3}{6\pi}. \quad (17)$$

Obviously, the following constraints on the new variables should be satisfied: $M_1 \geq 0$, $-\infty < M_2 < \infty$, $x, y \geq 0$, and $x + y \leq (M_1 - M_2)^2/4$. Here, the global minimum of function (17) is found in two stages. First, we find its minimum with respect to x and y varying in the closed and bounded domain $\omega = \{(x, y) : x, y \geq 0,\ x + y \leq (M_1 - M_2)^2/4\}$ (for fixed $M_{1,2}$). Then, the obtained expression is minimized with respect to the two remaining variables $M_{1,2}$.

Since $V(M_1, M_2, x, y)$ is a linear with respect to the x and y function, its smallest value in the triangular domain ω is reached at one of the vertices of this triangle, i.e., at one of the following points: $(x_1 = 0, y_1 = 0)$, $(x_2 = (M_1 - M_2)^2/4), y_2 = 0)$, and $(x_3 = 0, y_3 = (M_1 - M_2)^2/4)$. At these points the effective potential (17) takes the following values, respectively:

$$V_1(M_1, M_2) \equiv V(M_1, M_2, x = x_1, y = y_1) = \frac{g_1}{16}(M_1 - M_2)^2 + \frac{g_2}{16}(M_1 + M_2)^2 + \frac{M_1^3}{6\pi} + \frac{|M_2|^3}{6\pi}, \quad (18)$$

$$V_2(M_1, M_2) \equiv V(M_1, M_2, x = x_2, y = y_2) = \frac{h_1}{16}(M_1 - M_2)^2 + \frac{g_2}{16}(M_1 + M_2)^2 + \frac{M_1^3}{6\pi} + \frac{|M_2|^3}{6\pi}, \quad (19)$$

$$V_3(M_1, M_2) \equiv V(M_1, M_2, x = x_3, y = y_3) = \frac{h_2}{16}(M_1 - M_2)^2 + \frac{g_2}{16}(M_1 + M_2)^2 + \frac{M_1^3}{6\pi} + \frac{|M_2|^3}{6\pi}. \quad (20)$$

Obviously, in order to find the smallest value of the function $V(M_1, M_2, x, y)$ with respect to x and y, it is necessary to compare the quantities (18)–(20). For this purpose we set some fixed value of the coupling constant, g_1, and divide the plane of the coupling constants h_1 and h_2 into three domains I, II, and III (see Fig. 1 for the case $g_1 > 0$), where $I = \{(h_1, h_2) : h_1 > g_1, h_2 > g_1\}$, $II = \{(h_1, h_2) : h_1 < g_1, h_2 > h_1\}$, and $III = \{(h_1, h_2) : h_2 < g_1, h_2 < h_1\}$. The comparison of functions (18)–(20) yields the following: (i) in domain I the global minimum of the effective potential (17) with respect to the variables x and y has the form $(x = x_1, y = y_1)$, where $V(M_1, M_2, x, y)$ reaches its smallest value equal to $V_1(M_1, M_2)$ (18). (ii) If the coupling constants h_1 and h_2 lie in domain II, the smallest value of effective potential (17) with respect to the variables x and y is reached at the point $(x = x_2, y = y_2)$, where it is equal to $V_2(M_1, M_2)$ (19). (iii) Finally, if $(h_1, h_2) \in III$, the smallest value of the function $V(M_1, M_2, x, y)$ with respect to x and y is $V_3(M_1, M_2)$ reached at the point $(x = x_3, y = y_3)$. Let us now search the global minima of the functions $V_1(M_1, M_2)$, $V_2(M_1, M_2)$, and $V_3(M_1, M_2)$ with respect to the variables M_1 and M_2, whose admissible values are as follows: $M_1 \geq 0$, $-\infty < M_2 < \infty$.

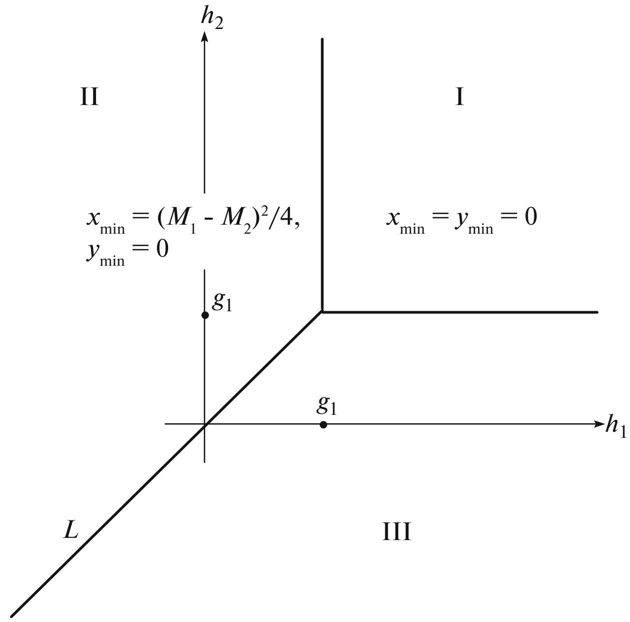

Fig. 1. The plane of the coupling constants h_1 and h_2 is divided into three domains I, II, and III. The coordinates x_{min} and y_{min} of the point of the smallest value of the effective potential $V(M_1, M_2, x, y)$ as a function of $(x, y) \in \omega = \{(x, y) : x, y \geq 0,\ x + y \leq (M_1 - M_2)^2/4\}$ are given in each domain. The line L is determined by the relation $L \equiv \{(h_1, h_2) : h_1 = h_2\}$. The figure corresponds to positive values of g_1.

Let us begin with the case $(h_1, h_2) \in I$, i.e., search the global minimum of the function $V_1(M_1, M_2)$. The system of stationarity equations for this function is

$$\frac{\partial V_1(M_1, M_2)}{\partial M_1} \equiv \frac{g_1}{8}(M_1 - M_2) + \frac{g_2}{8}(M_1 + M_2) + \frac{M_1^2}{2\pi} = 0, \quad (21)$$

$$\frac{\partial V_1(M_1, M_2)}{\partial M_2} \equiv \frac{g_1}{8}(M_2 - M_1) + \frac{g_2}{8}(M_1 + M_2) + \mathrm{sign}(M_2)\frac{M_2^2}{2\pi} = 0, \quad (22)$$

where sign(x) is the sign function. Let us denote the global minimum of the function $V_1(M_1, M_2)$ by (M_{10}, M_{20}). Obviously, it is the solution to the system of equations (21) and (22) and depends on the values of the coupling constants g_1 and g_2. By solving this system of stationarity equations, the behavior of the quantities M_{10} and M_{20} depending on g_1 and g_2 can be found (see Fig. 2 in which the plane of the coupling constants (g_1, g_2) is divided into three domains corresponding to the different form of M_{10} and M_{20}). Namely, if $g_{1,2} > 0$, then $M_{10} = M_{20} = 0$. If $g_2 < 0$ and $g_2 < g_1$, then $M_{10} = M_{20} = -\pi g_2/2$. IF $g_1 < 0$ and $g_2 > g_1$, then $M_{10} = -M_{20} = -\pi g_1/2$. Now that we know the expressions for $x_{\min}$ and $y_{\min}$ (see Fig. 1) in which M_1 and M_2 should be replaced by M_{10} and M_{20}, we return to the initial variables $\varphi_{1,2} \geq 0$ and $\sigma_{1,2} \geq 0$ and find the global minimum $(\sigma^o_{1,2}, \varphi^o_{1,2})$ of effective potential (15). The form of vacuum expectations $\langle\sigma_{1,2}\rangle \equiv \sigma^o_{1,2}$ and $\langle\varphi_{1,2}\rangle \equiv \varphi^o_{1,2}$ is found as a consequence. Thus, it can be easily demonstrated that in the case $(h_1, h_2) \in I$ and $g_{1,2} > 0$, i.e., if

$$(g_1, g_2, h_1, h_2) \in \Omega_{I0} \equiv \{(g_1, g_2, h_1, h_2): h_1 > g_1, h_2 > g_1; \quad g_1 > 0, g_2 > 0\}, \tag{23}$$

the coordinates of the global minimum of function (15) have the form $\sigma^o_{1,2} = 0$, $\varphi^o_{1,2} = 0$. This corresponds to zero vacuum averages of the scalar fields, $\langle\sigma_{1,2}\rangle = 0$ and $\langle\varphi_{1,2}\rangle = 0$. It can be demonstrated in a similar way that in the case $(h_1, h_2) \in I$, $g_1 < 0$, and $g_2 > g_1$, i.e., if

$$(g_1, g_2, h_1, h_2) \in \Omega_{I\sigma_1} \equiv \{(g_1, g_2, h_1, h_2): h_1 > g_1, h_2 > g_1; \quad g_1 < 0, g_2 > g_1\}, \tag{24}$$

we have $\langle\sigma_1\rangle = -\pi g_1/2$, $\langle\sigma_2\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$. Finally, if $(h_1, h_2) \in I$, $g_2 < 0$, and $g_2 < g_1$, i.e., if

$$(g_1, g_2, h_1, h_2) \in \Omega_{I\sigma_2} \equiv \{(g_1, g_2, h_1, h_2): h_1 > g_1, h_2 > g_1; \quad g_2 < 0, g_2 < g_1\}, \tag{25}$$

then $\langle\sigma_2\rangle = -\pi g_2/2$, $\langle\sigma_1\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$.

If the point (h_1, h_2) belongs to domain II (for $g_1 > 0$ see Fig. 1), it is necessary to search for the extremum of the function $V_2(M_1, M_2)$. The search for its global minimum (M_{10}, M_{20}) is considerably simplified, since the function $V_2(M_1, M_2)$ is obtained from (18) by replacing $g_1 \to h_1$. Taking this fact into account, the dependence of M_{10} and M_{20} on the coupling constants h_1 and g_2 can be easily found if the obvious replacement $g_1 \to h_1$ is made in Fig. 2. Now that $x_{\min}$ and $y_{\min}$ for $(h_1, h_2) \in II$ are known (see Fig. 1), the vacuum averages of the fields $\varphi_{1,2}$ and $\sigma_{1,2}$ can be easily obtained. Namely, if

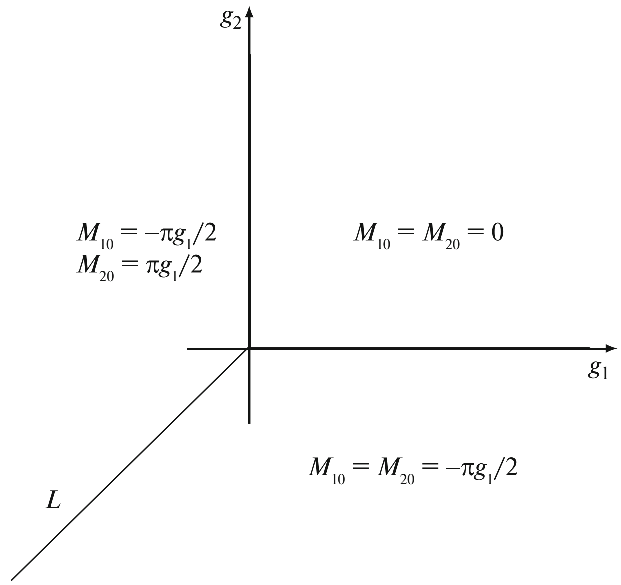

Fig. 2. The coordinates M_{10} and M_{20} of the global minimum of the function $V_I(M_1, M_2)$ (18) depending on the values of the coupling constants g_1, g_2. The line L is determined by the relation $L \equiv \{(g_1, g_2) : g_1 = g_2\}$.

$$(g_1, g_2, h_1, h_2) \in \Omega_{II0} \equiv \{(g_1, g_2, h_1, h_2): h_1 < g_1, h_2 > h_1; \quad h_1 > 0, g_2 > 0\}, \tag{26}$$

then $\langle\sigma_{1,2}\rangle = 0$ and $\langle\varphi_{1,2}\rangle = 0$. If

$$(g_1, g_2, h_1, h_2) \in \Omega_{II\varphi_1} \equiv \{(g_1, g_2, h_1, h_2): h_1 < g_1, h_2 > h_1; \quad h_1 < 0, g_2 > h_1\}, \tag{27}$$

we have $\langle\sigma_{1,2}\rangle = 0$, $\langle\varphi_1\rangle = -\pi h_1/2$, $\langle\varphi_2\rangle = 0$. If

$$(g_1, g_2, h_1, h_2) \in \Omega_{II\sigma_2} \equiv \{(g_1, g_2, h_1, h_2): h_1 < g_1, h_2 > h_1; \quad g_2 < 0, g_2 < h_1\}, \tag{28}$$

then $\langle\sigma_2\rangle = -\pi g_2/2$, $\langle\sigma_1\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$. The structure of the condensates $\langle\sigma_{1,2}\rangle$ and $\langle\varphi_{1,2}\rangle$ for the point (h_1, h_2) that lies in domain III can be easily found in a similar way. Therefore, if

$$(g_1, g_2, h_1, h_2) \in \Omega_{III0} \equiv \{(g_1, g_2, h_1, h_2): h_2 < g_1, h_2 < h_1; \quad h_2 > 0, g_2 > 0\}, \tag{29}$$

then $\langle\sigma_{1,2}\rangle = 0$ and $\langle\varphi_{1,2}\rangle = 0$. If

$$(g_1, g_2, h_1, h_2) \in \Omega_{III\varphi_2} \equiv \{(g_1, g_2, h_1, h_2): h_2 < g_1, h_2 < h_1; \quad h_2 < 0, g_2 > h_2\}, \tag{30}$$

we have $\langle\sigma_{1,2}\rangle = 0$, $\langle\varphi_2\rangle = -\pi h_2/2$, $\langle\varphi_1\rangle = 0$. Finally, if

$$(g_1, g_2, h_1, h_2) \in \Omega_{III\sigma_2} \equiv \{(g_1, g_2, h_1, h_2): h_2 < g_1, h_2 < h_1; \quad g_2 < 0, g_2 < h_2\}, \tag{31}$$

then $\langle\sigma_2\rangle = -\pi g_2/2$, $\langle\sigma_1\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$.

It follows from the above that the phase portrait of the initial model consists of five different phases which can be denoted by A, B, C, D, E.

Phase A occurs if the coupling constants (g_1, g_2, h_1, h_2) belong to the domain $\Omega_{I0} \cup \Omega_{II0} \cup \Omega_{III0}$ (see (23), (26), and (29)). In this domain, none of the discrete symmetries (4) is broken; all of the condensates of the boson fields and fermion masses are zero.

In phase B, i.e., for $(g_1, g_2, h_1, h_2) \in \Omega_{I\sigma_1}$ (see (24)), we have $\langle\sigma_1\rangle = -\pi g_1/2$, $\langle\sigma_2\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$. It can be seen, using field shift (16), that the fermion mass term $\sim \bar{\Psi}\Psi$ spontaneously appears in the Lagrangian; it is P invariant, but breaks both discrete chiral symmetries of the model Γ^3 and Γ^5 (4).

In phase C, i.e., for $(g_1, g_2, h_1, h_2) \in \Omega_{I\sigma_2} \cup \Omega_{II\sigma_2} \cup \Omega_{III\sigma_2}$ (see (25), (28), and (31)), we have $\langle\sigma_2\rangle = -\pi g_2/2$, $\langle\sigma_1\rangle = 0$, $\langle\varphi_{1,2}\rangle = 0$. Correspondingly, single-particle fermion excitations of the vacuum in this phase are described by the mass term of the form $\sim \bar{\Psi}\tau\Psi$ which breaks P parity but does not break Γ^3 and Γ^5 chiral symmetries of the model.

In phase D, i.e., for $(g_1, g_2, h_1, h_2) \in \Omega_{II\varphi_1}$ (see (27)), we have $\langle\sigma_{1,2}\rangle = 0$, $\langle\varphi_1\rangle = -\pi h_1/2$, $\langle\varphi_2\rangle = 0$. It can be easily seen that in this case both P parity and chiral Γ^5 symmetry (4) are broken, but Γ^3 invariance of the model holds. Single-fermion excitations are described by the mass term of the form $\sim \bar{\Psi}i\Gamma^5\Psi$.

Finally, **phase** E occurs if $(g_1, g_2, h_1, h_2) \in \Omega_{III\varphi_2}$ (see (30)). In this case $\langle\sigma_{1,2}\rangle = 0$, $\langle\varphi_2\rangle = -\pi h_2/2$, $\langle\varphi_1\rangle = 0$, and the ground state of the system is P and Γ^5 is invariant. Due to field shift (16), however, in phase E the mass term of the form $\sim \bar{\Psi}i\Gamma^3\Psi$ is spontaneously generated in the Lagrangian, which breaks the Γ^3 symmetry of the model.

CONCLUSIONS

It is well known that physical phenomena that occur in materials with a layered crystalline structure (high-temperature superconductivity, graphene, etc.) can be successfully described in the framework of relativistic (2+1)-dimensional models with four-fermion interaction of Gross–Neveu type. This is the reason that the interest in the investigation of effects that are predicted in the framework of these models, as well as the influence of various external factors on them, has not subsided during almost 30 years [3, 4, 25]. We note, however, that earlier (see, e.g., [20, 24]) these Gross–Neveu-type models were used, as a rule, to examine the simplest phase transition with the breaking of chiral invariance.

In this study we examined the phase structure of the generalized (2+1)-dimensional Gross–Neveu model (1) with several coupling constants that is P even and invariant with respect to two discrete chiral transformations, Γ^3 and Γ^5 (4). It was demonstrated that, depending on the values of the coupling constants, model (1) describes five different phases. In some of these either P parity (phase C), or chiral invariance (phases B and E) are spontaneously broken. A new phase D with a property that was unknown previously was found, namely, in this phase both P parity and chiral invariance are spontaneously broken. Therefore, it is possible to describe quite different phase transformations in the framework of one microscopic model (1).

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