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RADIOPHYSUCS, ELECTRONICS, ACOUSTICS

On the Influence of the Electron-Velocity Spread in a Beam on the Mechanism of Cherenkov Beam–Plasma Interaction

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Abstract—The instability of an electron beam in cold plasma is considered in the linear potential approximation with different velocity-distribution functions of beam electrons. It is demonstrated that the mechanism of beam instability in plasma changes as the electron-velocity spread is increased: the hydrodynamic single-particle instability mode evolves into the hydrodynamic collective mode or the single-particle kinetic one. Instability growth rates in different modes are determined analytically and numerically.

Keywords: Cherenkov effect, beam–plasma instabilities, dispersion equation, instability growth rate.

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INTRODUCTION

The electron-velocity spread in a beam may be neglected in the analysis of Cherenkov-beam instability in plasma if the following inequality is satisfied:

$$|\omega - \mathbf{k}\cdot\mathbf{u}| \gg k v_{Tb}, \quad (1)$$

where $\mathbf{u}$ is the velocity of the directed motion of beam electrons, v_{Tb} is the velocity spread ("thermal" velocity of beam electrons), ω and $\mathbf{k}$ are the frequency and the wave vector of a wave induced by the electron beam in plasma, and $k = |\mathbf{k}|$. Since $\omega \approx \mathbf{k}\cdot\mathbf{u}$ in the case of Cherenkov-beam instability, inequality (1) may be written as

$$|\delta\omega/\omega| \gg v_{Tb}/u. \quad (2)$$

Here, $\delta\omega$ is the growth rate of the instability and $u = |\mathbf{u}|$. Inequality (2) has a simple physical meaning: the displacement of a beam electron due to "thermal" motion during the development of instability is small compared to the wavelength. If inequalities (1) or (2) are satisfied the beam instability is called hydrodynamic; if the inequalities of the opposite sense are satisfied the beam instability is called kinetic. In fact, it will be shown below that the beam instability may be hydrodynamic, even if inequalities opposite to (1) or (2) are satisfied; the deciding factor here is the nature of the velocity-distribution function of beam electrons. In the present study, certain aspects of the linear theory of Cherenkov-beam instabilities in plasma in the case when inequalities (1) or (2) are not satisfied are considered. Much attention is paid to the dependence of the mechanism of beam instability in plasma on the velocity-distribution function of beam electrons.

A great number of papers focused on the study of beam instability in plasma have been published since the discovery of this effect in 1949 [1, 2]. The results of these studies were summarized in celebrated textbooks and monographs [3–6]. A large number of studies on the kinetic theory of beam–plasma instabilities have also been published [7–10]. Specifically, kinetic effects in the development of beam instability are taken into account in astrophysical applications [11]. Beam instabilities also manifest themselves in non-equilibrium gas discharge plasma [12, 13] and the corresponding characteristic thermal velocities may be of the same order of magnitude as the directed electron velocity. This poses the question of the reason that this seemingly closed issue is brought up again in the present study. The issue is that in our view, certain aspects of hydrodynamic and kinetic modes of beam–plasma interaction have still escaped the attention of researchers. Additional examination of the interaction between a beam and plasma in the case when inequalities (1) or (2) are not satisfied should also be of benefit to plasma microwave electronics [14], since the majority of experimental and theoretical studies concerning plasma microwave electronics rest on the assumption that inequalities (1) or (2) are well satisfied. This is in fact the case when plasma microwave radiators based on relativistic high-current electron beams that operate in centimeter and subcentimeter wavelength ranges are considered. Problems arise at shorter wavelengths: the instability growth rate may decrease[1] with an increase in frequency ω. As a result, inequality (2) may be violated.

[1] In the case of Cherenkov radiators based on surface (cable) plasma waves, the beam–plasma wave coupling weakens exponentially with an increase in frequency, thus changing the mode of beam instability in plasma: the single-particle (Compton) mode evolves into the collective (Raman) one. This results in a drastic reduction in the growth rate.

1. BASIC RELATIONSHIPS. THE CASE OF A SINGLE-VELOCITY BEAM

Let us consider a boundless cold electron plasma penetrated by a boundless nonrelativistic low-density electron beam. The coordinate axis z is directed along the vector of velocity of directed motion of the beam; i.e., $\mathbf{u} = \{0, 0, u\}$. Let us also limit ourselves to the analysis of potential waves propagating along axis z; i.e., $\mathbf{k} = \{0, 0, k\}$. The frequency spectrum $\omega(k)$ of waves in such a plasma–beam system is determined from the known dispersion equation [15, 16]

$$1 - \frac{\omega_p^2}{\theta^2} + \frac{\omega_b^2}{k}\int \frac{\partial f_{0b}(\mathbf{v})}{\partial v_z}\frac{1}{\omega - kv_z}d\mathbf{v} = 0. \quad (3)$$

Here, ω_p is the Langmuir frequency of plasma electrons, ω_b is the Langmuir frequency of beam electrons, $f_{0b}(\mathbf{v})$ is the undisturbed (and normalized to unity) velocity-distribution function of beam electrons $\mathbf{v} = \{v_x, v_y, v_z\}$, and $d\mathbf{v} = dv_x dv_y dv_z$. Note that the velocity spread of plasma electrons is easy to factor in, but there is no need to do it here. In contrast to the beam, the condition under which the thermal motion of plasma electrons becomes negligible is written as $|\omega| \gg kv_{Tp}$, where v_{Tp} is the thermal velocity of plasma electrons. In the case of Cherenkov-beam instability in plasma, this condition is reduced to the $v_{Tp}/u \ll 1$ inequality, which is easy to satisfy. In accordance with the Landau rule [17], the integration in dv_z in Eq. (3) is performed along a contour in complex plane $v_z = v'_z + iv''_z$ that goes around singular point ωk from below. The low density of beam electrons implies that the following inequality is satisfied:

$$\omega_b^2 \ll \omega_p^2. \quad (4)$$

In the case of a single-velocity electron beam with distribution function

$$f_{0b}(\mathbf{v}) = \delta(v_x)\delta(v_y)\delta(v_z - u), \quad (5)$$

dispersion equation (3) is reduced to the following form [15]:

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - ku)^2} = 0. \quad (6)$$

If inequality (4) is satisfied, the solutions of dispersion equation (6) are specified by the known formulas [15] (for definiteness, it is assumed that $k > 0$)

$$ku \neq \omega_p: \begin{cases} \omega_{1,2} = ku \pm \begin{cases} i\dfrac{\omega_b}{\sqrt{\omega_p^2 - k^2u^2}}ku, & ku < \omega_p, \\ i\dfrac{\omega_b}{\sqrt{k^2u^2 - \omega_p^2}}ku, & ku > \omega_p, \end{cases} \\ \omega_3 = \omega_p, \quad \omega_4 = -\omega_p \end{cases} \quad (7)$$

and

$$ku = \omega_p: \begin{cases} \omega_{1,2} = ku + \left(-\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\omega_b^2}{2\omega_p^2}\right)^{1/3}\omega_p, \\ \omega_3 = ku + \left(\dfrac{\omega_b^2}{2\omega_p^2}\right)^{1/3}\omega_p, \quad \omega_4 = -\omega_p. \end{cases} \quad (8)$$

Formulas (8) characterize the resonance Cherenkov-beam instability in plasma ($\mathrm{Im}\,\omega_1 > 0$). It is also called the single-particle induced Cherenkov-effect instability [18].[2] Formulas (7) at $ku < \omega_p$ characterize the aperiodic (nonresonance) beam instability in plasma. According to the known classification [19], it belongs to the negative mass instability type.

Dispersion equation (6) may be rewritten in the form

$$(\omega^2 - \omega_p^2)[(\omega - ku)^2 - \omega_b^2] = \omega_p^2\omega_b^2. \quad (9)$$

Two interacting subsystems (the beam subsystem with dispersion equation $(\omega - ku)^2 - \omega_b^2 = 0$ and the plasma subsystem with dispersion equation $\omega^2 - \omega_p^2 = 0$) stand out in (9). At the points of resonance of beam and plasma waves the solution of Eq. (9) should be sought in the form

$$\omega = \omega_p + \delta\omega = ku \pm \omega_b + \delta\omega, \quad (10)$$

where $\delta\omega$ is a frequency correction associated with the wave interaction. It turns out that $\omega_b \ll |\delta\omega|$ if inequality (4) is satisfied. Therefore, the wave resonance condition is reduced to the Cherenkov resonance condition $\omega = ku$, and $\delta\omega$ is taken from formulas (8). Thus, only the single-particle induced Cherenkov effect is feasible in the case of a single-velocity low-density beam (4). It will be shown below that the collective induced Cherenkov effect is also feasible if a velocity spread of beam electrons occurs.

2. THE BEAM INSTABILITY IN THE HYDRODYNAMIC APPROXIMATION WITH THE GAS-KINETIC PRESSURE FACTORED IN

Let us set the velocity-distribution function of beam electrons in the following form:

$$f_{0b}(\mathbf{v}) = \delta(v_x)\delta(v_y)\Delta(v_z - u),$$
$$\Delta(v_z - u) = (2v_{Tb})^{-1}\begin{cases} 1, & v_z \in [u - v_{Tb}, u + v_{Tb}], \\ 0, & v_z \notin [u - v_{Tb}, u + v_{Tb}]. \end{cases} \quad (11)$$

Inserting distribution (11) into Eq. (3) and performing elementary integration, one obtains the following dispersion equation:

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - ku)^2 - k^2v_{Tb}^2} = 0. \quad (12)$$

[2] In the foreign literature, this instability is also called the induced Compton effect.

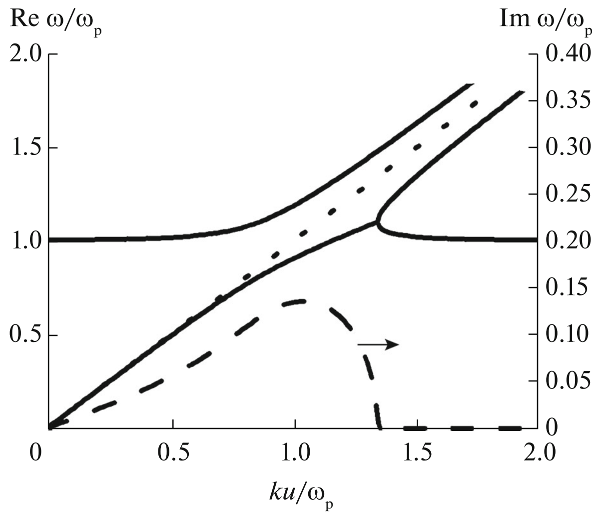

Fig. 1. Dispersion curves of the beam–plasma system with the gas-kinetic pressure in the beam factored in at $v_{Tb}/u = 0.02$: the real part of the frequency (solid curves, left axis) and instability growth rate (dashed curve, right axis). The dotted straight line is $\omega = ku$.

At $v_{Tb} = 0$, Eq. (12) turns into Eq. (6) and the effects associated with the velocity spread of beam electrons disappear. Let us rewrite Eq. (12) in the form similar to (9):

$$(\omega^2 - \omega_p^2)[(\omega - ku)^2 - k^2 v_{Tb}^2 - \omega_p^2] = \omega_p^2 \omega_b^2. \quad (13)$$

Equation (13) allows one to consider the beam–plasma instability as an instability in the system of coupled waves of plasma and a beam. The right-hand-side of (13) may be regarded as the coupling coefficient. If such coupling is lacking ($\omega_b \to 0$), Eq. (13) is reduced to a product of two independent factors. The first factor equated to zero defines the spectrum of plasma waves; the second one defines the spectrum of beam waves. In the case of the resonance of plasma and beam waves when ω and k satisfy system

$$\begin{cases} \omega = \omega_p \\ \omega = ku \pm \sqrt{k^2 v_{Tb}^2 + \omega_b^2}, \end{cases} \quad (14)$$

the solution of Eq. (13) should be sought in the form $\omega = \omega_p + \delta\omega$ (see (10)). On inserting this into (13), one obtains the following equation:

$$\delta\omega^2(\delta\omega \pm 2\sqrt{k^2 v_{Tb}^2 + \omega_b^2}) = \omega_b^2 \omega_p / 2. \quad (15)$$

The upper sign in (15) corresponds to the resonance of a plasma wave with a fast beam wave; the lower one, to the resonance of a plasma wave with a slow beam wave. If inequality

$$|\delta\omega| \gg \sqrt{k^2 v_{Tb}^2 + \omega_b^2} \quad (16)$$

is satisfied, the solutions of Eq. (15) $\delta\omega_{1,2,3}$ are given in formulas (8) (the corrections to the principal term ku in the expressions for $\omega_{1,2,3}$). If the inequality opposite to (16) is satisfied, instability emerges only in the case of resonance of a plasma wave with a slow beam wave (the lower sign in Eq. (15)). Then

$$\delta\omega_{1,2} = \pm i \frac{\omega_b \omega_p^{1/2}}{2(k^2 v_{Tb}^2 + \omega_b^2)^{1/4}}. \quad (17)$$

The instability with growth rate $\delta\omega_1$ corresponds to the collective induced Cherenkov-effect instability [18].[3] The condition of applicability of solution (17), which is the inequality opposite to (16), is reduced to condition

$$\omega_b^2 \omega_p \ll (k^2 v_{Tb}^2 + \omega_b^2)^{3/2}. \quad (18)$$

If the electron-velocity spread in a beam is lacking, condition (18) and inequality (4) may not be satisfied simultaneously. However, if the velocity spread is considerable, (18) is reduced to inequality $\omega_b^2 \omega_p \ll k^3 v_{Tb}^2$, which is satisfied easily. Thus, the mechanism of resonance Cherenkov-beam instability in plasma changes as the plasma electron-velocity spread is increased: the single-particle instability mode evolves into the collective mode.

The results of solving dispersion equation (13) numerically are shown in Figs. 1 and 2. The following ratio of densities of the electron beam and plasma was used here and below in the calculations: $\omega_b^2/\omega_p^2 = 0.01$. In Fig. 1, $v_{Tb}/u = 0.02$; in Fig. 2, $v_{Tb}/u = 0.2$. The solid curves in these figures represent real values of frequency (scale on the left vertical axis); the dashed curves represent the imaginary components of the frequency (instability growth rates; scale on the right axis). The dotted straight line is the Cherenkov resonance line $\omega = ku$. Figure 1 differs fundamentally from Fig. 2. The value of $v_{Tb}/u = 0.02$ corresponds to an almost single-velocity beam: since inequality (16), although not strong, is satisfied, the single-particle induced Cherenkov effect is observed at resonance. The instability region extends from $k = 0$ to a value somewhat higher than $k = \omega_p/u$. The dispersion curves in Fig. 1 agree well with formulas (7) and (8). The structure of the dispersion curves in Fig. 2 is qualitatively different: the instability region contracts, since long-wavelength disturbances are stable. At $v_{Tb}/u = 0.2$, the inequality opposite to (16) is satisfied. As a result, the collective induced Cherenkov effect, which affects the structure of dispersion curves, is observed. The maximum growth rate in Fig. 2 agrees fairly well with (17).

The instabilities considered above are hydrodynamic ones. This is also true of the instability with growth rate (17), although the development of this instability requires a fairly large electron-velocity spread in the beam. The issue is that the integral over dv_z in dispersion equation (3) is a complex one, owing to the occurrence of a singular point. Therefore, dispersion equation (3) may also incorporate an imaginary part in an explicit form. The instability may be regarded as the kinetic type only when its growth rate is derived from the imaginary part

[3] In the foreign literature, this instability is also called the induced Raman effect.

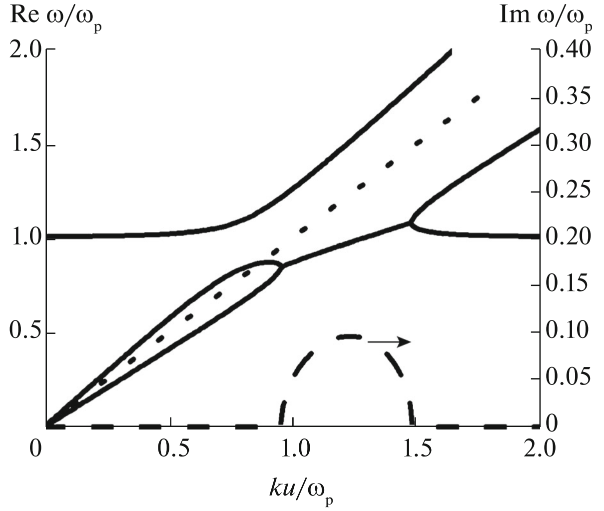

Fig. 2. The dispersion curves of the beam–plasma system with the gas-kinetic pressure in the beam factored in at $v_{Tb}/u = 0.2$: the real part of the frequency (solid curves, left axis) and instability growth rate (dashed curve, right axis). The dotted straight line is $\omega = ku$.

of the dispersion equation.[4] In the contrary case, the instability should be regarded as a hydrodynamic one. It is evident that the singular point produces no contribution to the integral in the case of distribution (11). As a result, dispersion equation (12) turns into an algebraic equation with real coefficients.

One may also arrive at dispersion equation (12) using the following velocity-distribution function of beam electrons:

$$f_{0b}(\mathbf{v}) = \frac{m}{4\pi v_{Tb}}\delta\left(\frac{m[v_x^2 + v_y^2 + (v_z - u)^2]}{2} - \frac{mv_{Tb}^2}{2}\right). \quad (19)$$

Distribution function (19) corresponds to the case when the beam is produced by accelerating isotropically distributed electrons with the same energy $mv_{Tb}^2/2$. Dispersion equation (12) is also obtained in the hydrodynamic model when hydrodynamic equations with gas-kinetic pressure $P = \rho v_{Tb}^2$, where ρ is the electron gas density, are used to characterize beam electrons.

3. THE INSTABILITY OF A BEAM WITH A MAXWELLIAN VELOCITY-DISTRIBUTION FUNCTION OF ELECTRONS

Let us now consider an electron beam with a "shifted" Maxwellian velocity-distribution function of electrons

$$f_{0b}(\mathbf{v}) = (2\pi v_{Tb}^3)^{-3/2}\exp\left(\frac{v_x^2 + v_y^2 + (v_z - u)^2}{2v_{Tb}^2}\right). \quad (20)$$

Inserting distribution (20) into general equation (3), one obtains the following dispersion equation:

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{k^2 v_{Tb}^2}\left(1 - J_+\left(\frac{\omega - ku}{kv_{Tb}}\right)\right) = 0. \quad (21)$$

Here, function [15, 20]

$$J_+(x) = \frac{x}{\sqrt{2\pi}}\int_C \frac{\exp(-\xi^2/2)}{x - \xi}d\xi \quad (22)$$

was introduced, and contour C starts and ends at real axis ξ at $\xi = \pm\infty$ and runs below singular point $\xi = x$.

Let us examine the transition to the single-velocity electron beam limit ($v_{Tb} \to 0$) using dispersion equation (21). The known asymptotic expansion [15, 20] yields the following at $|x| \gg 1$ and $\mathrm{Im}x > 0$:

$$J_+(x) \approx 1 + 1/x^2. \quad (23)$$

If $\mathrm{Im}x < 0$, formula (23) is valid only when $|\mathrm{Re}x| > |\mathrm{Im}x|$. Keeping formula (23) in mind, one can readily see that dispersion equation (21) transforms into Eq. (6) if inequality (1) is satisfied. However, not all solutions of Eq. (6) may be derived from the solutions of Eq. (21) by passing to the single–velocity beam limit. No difficulties arise in the case of solutions with positive and zero imaginary parts ($\omega_{1, 3, 4}$ in formulas (7) and (8)): these solutions are derived from Eq. (21) at $v_{Tb} \to 0$ (see below). In order for solution ω_1 from formulas (8) to be applicable, the following inequality should be satisfied (see (2)):

$$v_{Tb}/u \ll (\omega_b^2/2\omega_p^2)^{1/3}. \quad (24)$$

The condition of applicability of solution ω_1 from formulas (7) is more stringent:

$$v_{Tb}/u \ll \omega_b/\omega_p. \quad (25)$$

The solutions with a negative imaginary part (ω_2 in formulas (7) and (8)) do not follow from Eq. (21) in the $v_{Tb} \to 0$ limit. In fact, if one assumes that $x = (\omega^2 - ku)/kv_{Tb}$, $|\mathrm{Re}x| < |\mathrm{Im}x|$; therefore, asymptotic formula (23) does not hold true. Thus, no problems arise in the examination of solutions of dispersion equations that characterize waves that grow under beam instability; only the solutions that characterize decaying waves are problematic. It may seem that the theoretical problems related to decaying waves are not worth discussing. This is not quite true, since all waves (both decaying and growing) need to be taken into account correctly in the process of solving initial-value and boundary problems in order to satisfy the initial and boundary conditions [21]. It is evident that a definite inconsistency associated with the transition from the kinetic beam treatment to the hydrodynamic one warrants a separate study, but this is outside the scope of the present work.

Let the inequality opposite to (1) be satisfied now. Using asymptotic expansion [15, 20]

$$J_+(x) \approx -i\sqrt{\pi 2}x, \quad |x| \ll 1, \quad (26)$$

[4] It should not be confused with dissipative instabilities such as the instability of a system with negative friction.

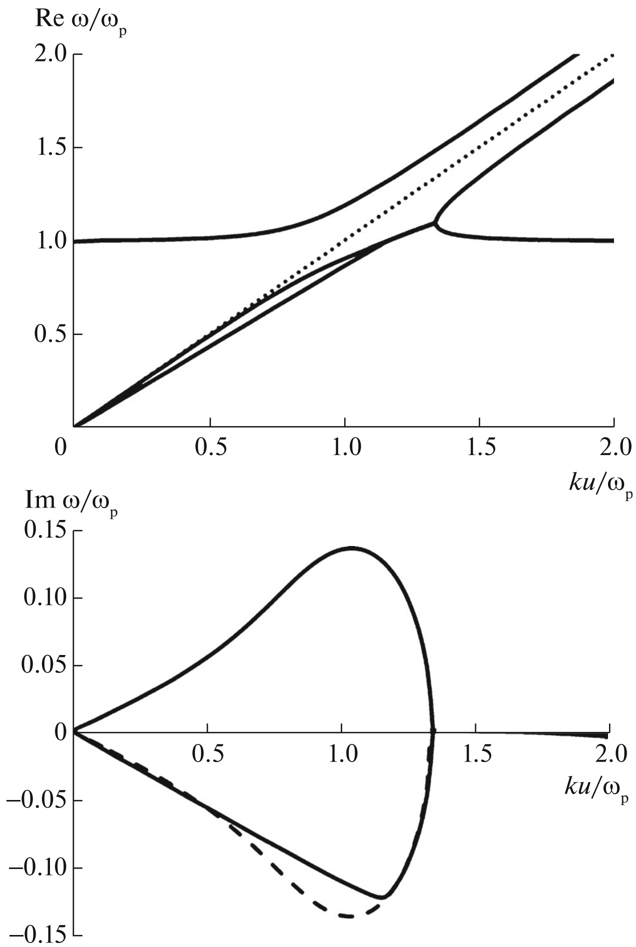

Fig. 3. The dispersion curves of the beam–plasma system with a Maxwellian velocity-distribution function of beam electrons at $v_{Tb}/u = 0.02$: the real part of the frequency (upper panel) and the imaginary part of the frequency (lower panel). The dotted straight line is $\omega = ku$.

we then transform dispersion equation (21) to the following form:

$$1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_b^2}{k^2 v_{Tb}^2} + i\sqrt{\frac{\pi}{2}}\frac{\omega_b^2(\omega - ku)}{k^3 v_{Tb}^3} = 0. \quad (27)$$

Solving Eq. (27) by successive approximations, we obtain $\omega = \omega_0 + \delta\omega$, where

$$\omega_0^2 = \frac{\omega_p^2}{1 + \omega_b^2/(k^3 v_{Tb}^3)}, \quad \delta\omega = -i\sqrt{\frac{\pi}{8}}(\omega_0 - ku)\frac{\omega_b^2\omega_0^3}{\omega_p^2 k^3 v_{Tb}^3}. \quad (28)$$

It can be seen that the imaginary part of the frequency is positive at $ku > \omega_0$ (i.e., beam instability is present). This instability is kinetic, since dispersion equation (27) is, unlike, e.g., Eqs. (6) and (12), an algebraic one with complex coefficients.

Equation (27) and formulas (28) are applicable only if the inequalities opposite to (1) and (2) are satisfied. In the general case, it is not possible to obtain analytical results. Therefore, let us analyze the numerical solutions of Eq. (21) for $\omega_b^2/\omega_p^2 = 0.01$ with varying v_{Tb}/u ratios. Figure 3 shows the numerical solutions of dispersion equation (21) at a small value of v_{Tb}/u. These solutions agree with limiting solutions $\omega_{1,2,3}$ in formulas (7) and (8). The growth rate is maximized under the resonance condition $\omega_p \approx ku$ and corresponds to the value given by formulas (8). In general, the numerical results at $v_{Tb}/u = 0.02$ match the analytical ones in the cold electron beam limit. The sole exception is the branch with $\mathrm{Im}\omega < 0$, which corresponds to decaying oscillations (the dashed curve in Fig. 3 represents the solution of dispersion equation (6) with $\mathrm{Im}\omega < 0$). This distinction persists at an arbitrarily small difference between v_{Tb}/u and zero and is attributed to the fact that asymptotic expansion (23) may not be used at $\mathrm{Im}x < 0$ and $|\mathrm{Re}x| < |\mathrm{Im}x|$.

As the velocity spread in a beam is increased further, dispersion equation (6) becomes inapplicable (starting from $v_{Tb}/u = 0.1$). Figure 4 shows the dependences of the instability growth rate on the wave number for various values of v_{Tb}/u. As the velocity spread is increased, the maximum growth rate decreases and shifts toward shorter wavelengths, while the instability region expands and also shifts toward larger wave numbers. At small values of ku/ω_p the instability vanishes.

4. INSTABILITIES OF A BEAM WITH MODIFIED MAXWELLIAN VELOCITY-DISTRIBUTION FUNCTIONS OF ELECTRONS

Adequate data on the velocity-distribution function of beam electrons are difficult to retrieve from experimental studies focused on plasma microwave electronics. However, such data are essential for further development of the theory of beam instabilities in plasma. The fact is that complex frequencies of waves in the beam–plasma system depend strongly on the nature of the distribution function of beam electrons. This is confirmed by the results described above and by the following two examples. Let us first consider an electron beam with a "half-Maxwellian" velocity-distribution function of electrons

$$f_{0b}(\mathbf{v}) = 2(2\pi v_{Tb}^2)^{-3/2}\exp\left(-\frac{v_x^2 + v_y^2 + (v_z - u)^2}{2v_{Tb}^2}\right)\theta(v_z - u), \quad (29)$$

where $\theta(\xi)$ is the Heaviside step function (its value is one at $\xi > 1$ and zero in the contrary case). The free path length of beam electrons in systems based on the interaction of an accelerated electron stream with plasma may turn out to be larger than the characteristic longitudinal size of the system. The collisionless electron transit mode is then established and it is fair to assume that the electron distribution function takes the form (29). Certain questions related to the development of beam instabilities in space (specifically, the cases when the distribution function has step-like singularities) were discussed in [22].

Inserting function (29) into dispersion equation (3) and performing calculations similar to the ones in the process of derivation of Eq. (21), we obtain the following dispersion equation:

$$1-\frac{\omega_p^2}{\omega^2}+\frac{\omega_b^2}{k^2 v_{Tb}^2}\left(1-\sqrt{\frac{2}{\pi}}x\int_0^\infty\frac{\exp(-\xi^2/2)}{x-\xi}d\xi+\sqrt{\frac{2}{\pi}}\frac{1}{x}\right)0, \quad (30)$$

where $x = (\omega - ku)(kv_{Tb})$, and integration is performed along a contour that goes along the positive semiaxis and goes around the singular point $\xi = x$ from below. Transforming the integral in Eq. (30), we get

$$\sqrt{\frac{2}{\pi}}x\int_0^\infty\frac{\exp(-\xi^2/2)}{x-\xi}d\xi=\sqrt{\frac{2}{\pi}}x^2\int_0^\infty\frac{\exp(-\xi^2/2)}{x^2-\xi^2}d\xi + \sqrt{\frac{2}{\pi}}x\int_0^\infty\frac{\xi\exp(-\xi^2/2)}{x^2-\xi^2}d\xi. \quad (31)$$

It is easy to verify that the first term at the right-hand side of relation (31) is the function $J_+(x)$. After the change of the variable in integration is performed, the second term may be expressed in terms of the integral exponential function

$$\mathrm{Ei}(x) = -\mathrm{V.p.}\int_{-x}^\infty \xi^{-1}\exp(-\xi)d\xi. \quad (32)$$

As a result, dispersion equation (30) is rewritten as

$$1-\frac{\omega_p^2}{\omega^2}+\frac{\omega_b^2}{k^2 v_{Tb}^2}\left(1-J_+(x)-\frac{x\exp(-x^2/2)}{\sqrt{2\pi}} \times(\mathrm{Ei}(x^2/2)-i\pi)+\sqrt{\frac{2}{\pi}}\frac{1}{x}\right)=0. \quad (33)$$

At large values of $|x|$ (i.e., in the single-velocity beam limit (1)), dispersion equation (33) transforms into Eq. (6). In the opposite limiting case, the last term in round brackets in (33) is the dominant one and the dispersion equation takes the form

$$1-\frac{\omega_p^2}{\omega^2}+\sqrt{\frac{2}{\pi}}\frac{\omega_b^2}{(\omega-ku)kv_{Tb}}=0. \quad (34)$$

Equation (34) may be rewritten so as to emphasize the interaction between beam and plasma waves:

$$(\omega^2-\omega_p^2)\left(\omega-ku+\sqrt{\frac{2}{\pi}}\frac{\omega_p^2}{kv_{Tb}}\right)=-\omega_p^2\sqrt{\frac{2}{\pi}}\frac{\omega_b^2}{kv_{Tb}}. \quad (35)$$

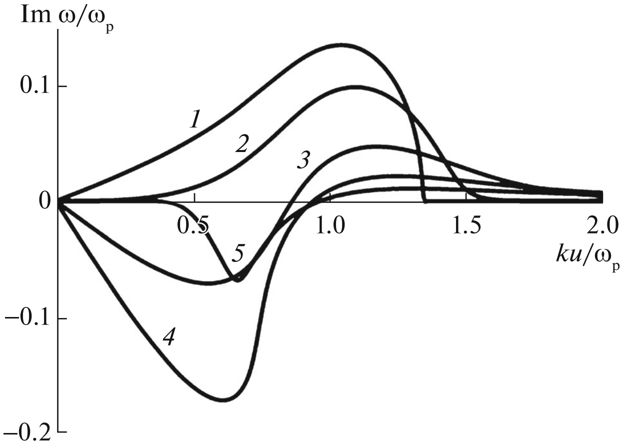

Fig. 4. The growth rate of the beam instability of a system with a Maxwellian velocity-distribution function of beam electrons at various values of v_{Tb}/u: 0.02 (curve *1*), 0.1 (*2*), 0.2 (*3*), 0.3 (*4*), and 0.4 (*5*).

The solutions of Eq. (35) that characterize waves that grow in the presence of instability should be sought in the form

$$\omega = \omega_p + \delta\omega, \quad |\delta\omega| \ll \omega_p. \quad (36)$$

The inequality in (36) implies that it is not some complex beam—plasma excitation that occurs, but rather a plasma wave is excited (emitted) by a beam in the presence of instability. In terms of the coupled-wave theory, this inequality implies that the coupling between a plasma wave and a beam wave is weak [21, 23].

Inserting (36) into Eq. (35), we transform (35) to the form

$$\delta\omega(\delta\omega+\Delta)+\sqrt{\frac{1}{2\pi}}\omega_p\frac{\omega_p^2}{kv_{Tb}}=0, \quad (37)$$

where

$$\Delta=\omega_p-ku+\sqrt{\frac{2}{\pi}}\frac{\omega_b^2}{kv_{Tb}} \quad (38)$$

is the wave resonance detuning. It follows from (37) that

$$\delta\omega=-\frac{\Delta}{2}\pm\sqrt{\frac{\Delta^2}{4}-\sqrt{\frac{1}{2\pi}}\omega_p\frac{\omega_b^2}{kv_{Tb}}}. \quad (39)$$

The maximum growth rate

$$\delta\omega=i\frac{\omega_b}{\sqrt[4]{2\pi}}\left(\frac{\omega_p}{kv_{Tb}}\right)^{1/2} \quad (40)$$

corresponds to zero detuning, when the inequality in (36) is reduced to the following:

$$\frac{\omega_b}{\omega_p}\ll\left(\frac{kv_{Tb}}{\omega_p}\right)^{1/2}\approx\left(\frac{v_{Tb}}{u}\right)^{1/2}. \quad (41)$$

Inequality (4) was used to obtain estimate (41).

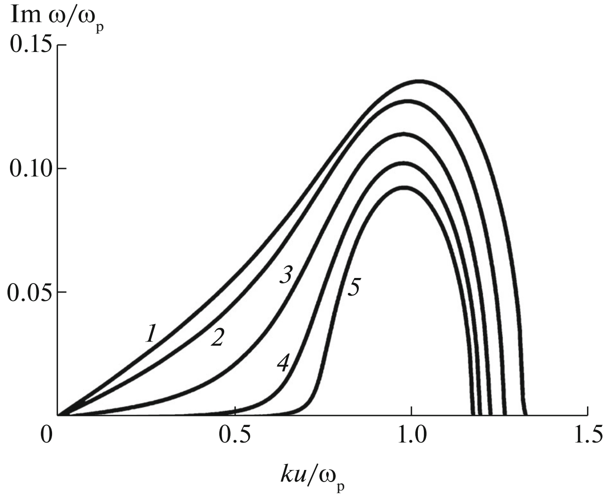

Fig. 5. The growth rate of the beam instability of a system with a half-Maxwellian velocity-distribution function of beam electrons at various values of v_{Tb}/u: 0.02 (curve *1*), 0.1 (*2*), 0.2 (*3*), 0.3 (*4*), and 0.4 (*5*).

According to the adopted classification, the instability with growth rates (39) and (40) is a hydrodynamic one. This is due to the fact that the beam wave $\omega = ku - \sqrt{2\pi^{-1}}\omega_b^2(kv_{Tb})$, which interacts with a plasma wave, has negative energy. At the same time, it should be noted that the beam wave was characterized within the kinetic approach.

The results of solving dispersion equation (33) numerically for $\omega_b^2/\omega_p^2 = 0.01$ are shown in Fig. 5. At $v_{Tb}/u = 0.02$, the instability growth rate (curve *1*) is almost the same as the rate that corresponds to the interaction of a cold beam with plasma. As the electron-velocity spread in the beam is increased the instability region contracts and the growth rate maximum becomes slightly lower although its position remains almost the same. At a fairly large spread $v_{Tb}/u > 0.4$, the numerical results agree with the ones that are obtained using analytical formulas (39) and (40).

Finally, let us consider the development of beam–plasma instabilities in collisionless systems where an electron beam emitted from the cathode surface is accelerated by an applied potential difference. The emitted beam has the distribution function

$$f_{0b}(\mathbf{v}) = A\exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2v_{Tb}^2}\right)\theta(v_z). \qquad (42)$$

The normalization constant in this function will be determined later based on the beam parameters in the space of interaction with a plasma wave. Having passed through an accelerating gap with potential difference V, an electron acquires additional energy $mu^2/2 = eV$ and the distribution function is transformed to the form

$$f_{0b}(\mathbf{v}) = A\exp\left(-\frac{v_x^2 + v_y^2 + v_z^2 - u^2}{2v_{Tb}^2}\right)\theta(v_z - u). \qquad (43)$$

In real plasma microwave electronic systems the $v_{Tb} \ll u$ relation is satisfied. Therefore, the distribution function is localized near $v_z = u$, thus providing an opportunity to rewrite (43) in the following approximate form:

$$f_{0b}(\mathbf{v}) = A\exp\left(-\frac{v_x^2 + v_y^2 + 2u(v_z - u)}{2v_{Tb}^2}\right)\theta(v_z - u). \qquad (44)$$

Assuming that the electron density in the beam in the region of beam–plasma wave interaction is n_b, we determine the normalization constant A and write the distribution function of beam electrons in the final form:

$$f_{0b}(\mathbf{v}) = \sqrt{2\pi}\frac{u}{v_{Tb}}(2\pi v_{Tb}^2)^{-3/2} \times\exp\left(-\frac{v_x^2 + v_y^2 + 2u(v_z - u)}{2v_{Tb}^2}\right)\theta(v_z - u). \qquad (45)$$

Inserting function (45) into dispersion equation (3) and performing calculations similar to those carried out in order to derive Eq. (30), we obtain the following dispersion equation:

$$1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_b^2}{k^2 v_{Tb}^2}\frac{u^2}{v_{Tb}^2}\left(-e^{-x}(\mathrm{Ei}(x) - i\pi) + \frac{1}{x}\right) = 0, \qquad (46)$$

where $x = (\omega - ku)u/(kv_{Tb}^2)$, and function $\mathrm{Ei}(x)$ is defined by relation (32).

In the single-velocity beam limit (i.e., at large $|x|$), dispersion equation (46) transforms into Eq. (6). In the opposite limiting case, the $1/x$ term in (46) is the dominant one and the dispersion equation takes the form

$$1 - \frac{\omega_p^2}{\omega^2} + \frac{u}{v_{Tb}}\frac{\omega_b^2}{(\omega - ku)kv_{Tb}} = 0. \qquad (47)$$

Equation (47) is the same as Eq. (34), which was analyzed above, except for the notation. The difference consists in the effective increase in the beam density due to the factor u/v_{Tb}. Proceeding as before, we obtain the following instability growth rate under conditions (36):

$$\delta\omega = -\frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} - \omega_p\frac{u}{2v_{Tb}}\frac{\omega_b^2}{kv_{Tb}}}, \qquad (48)$$

where the detuning Δ is

$$\Delta = \omega_p - ku\frac{u}{v_{Tb}}\frac{\omega_b^2}{kv_{Tb}}. \qquad (49)$$

The maximum growth rate (obtained at $\Delta = 0$) is

$$\delta\omega = i\omega_b \left(\frac{u}{2\nu_{Tb}} \frac{\omega_p}{k\nu_{Tb}}\right)^{1/2}. \quad (50)$$

Condition (36) is then reduced to a more stringent inequality

$$\frac{\omega_b}{\omega_p} \ll \frac{\nu_{Tb}}{u}. \quad (51)$$

The comparison of dispersion equations (34) and (47) and growth rates (40) and (50) shows that thermal effects manifest themselves at a lower electron-beam density in the case of a beam with distribution function (45).

CONCLUSIONS

The effect of the electron-velocity spread in a beam was analyzed in the context of its instability in plasma. The single-velocity electron-beam approximation, which holds true if condition (2) is satisfied, is traditional for plasma microwave electronics. If condition (2) is violated, the dissimilarity between the electron distribution function and the δfunction needs to be taken into account. Since the exact nature of the distribution function is often not known, we performed comparative analysis of different distribution functions constructed on simple physical grounds. Specifically, the description of a single-velocity electron beam was modified by the introduction of the gas-kinetic pressure. In addition, a completely thermalized electron beam with a Maxwellian distribution function and a beam with modified Maxwellian electron velocity-distribution functions were examined. The latter approximation is better suited for describing collisionless beam–plasma systems of plasma microwave electronics.

Two beam instability modes (single-particle and collective) may be distinguished in all the considered approximations. The single-particle instability mode is established at a small electron-velocity spread in a beam and is characterized by a wide region of wave numbers (from zero to a value around $\omega_p u$) where instability is manifested. As the velocity spread is increased, the instability region localizes near the resonance weave number ($\sim\omega_p u$), long-wavelength disturbances stabilize, and the instability mode changes to the collective one. The instability growth rate decreases as the electron-velocity spread in a beam is increased. However, the dependences of the instability growth rate on system parameters differ from one distribution function to another. As an example, the instability growth rate in a completely thermalized electron beam decreases with temperature much faster than in a beam with a half-Maxwellian distribution. More importantly, the mechanisms of instability of beams with Maxwellian and half-Maxwellian distributions at a large electron-velocity spread differ considerably. In the former case, the instability is kinetic (wave–particle interaction); in the latter case, the instability mechanism is the collective wave–wave interaction.

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