ISSN 0027-1349, Moscow University Physics Bulletin, 2017, Vol. 72, No. 1, pp. 16–22. © Allerton Press, Inc., 2017.
Original Russian Text © A.N. Bogolyubov, A.I. Erokhin, I.E. Mogilevsky, M.I. Svetkin, 2017, published in Vestnik Moskovskogo Universiteta, Seriya 3: Fizika, Astronomiya, 2017, No. 1, pp. 17–22.

THEORETICAL AND MATHEMATICAL PHYSICS

A Hybrid Method for the Solution of the Poisson Equation in the Domain of Metal-Dielectric Corners

A. N. Bogolyubov, A. I. Erokhin, I. E. Mogilevsky, and M. I. Svetkin*

Faculty of Physics, Moscow State University, Moscow, Russia

**e-mail: mihail-svetkin@mail.ru*

Received June 15, 2016; in final form, June 27, 2016

Abstract—The Poisson's equation in a two-dimensional domain that contains star metal-dielectric corners is considered. The corners can be only metal or only dielectric. In their neighborhood, the solution is bounded, but the gradient has power-law singularities. A numerical algorithm for solving this problem is proposed. The algorithm is based on the finite-element method and takes the asymptotic behavior of the solution in the neighborhood of metal-dielectric corners into account.

Keywords: Poisson equation, metal-dielectric corner, finite-element method.

DOI: 10.3103/S0027134916050064

INTRODUCTION

In many problems of electrostatics and electrodynamics, it is necessary to find an electromagnetic field in complex domains that contain metal or dielectric reentrant corners. The calculation of the field distribution inside the multigap waveguide system filled with dielectrics with different dielectric constants can be cited as an example of such problem.

The presence of an electromagnetic field singularity, which consists in the unlimited growth of some of its components close to the metal reentrant corner, has been proven theoretically [1–5] and experimentally [6–8]. The field singularity also takes place in the presence of dielectric corners [9]. In this paper the case when the domain contains a star metal-dielectric corner that also leads to the emergence of the field singularity was considered.

Various numerical methods are used for the calculation of such problems [10–15]. However, the presence of singularities significantly impairs their convergence speed and accuracy. One of the most common ways to increase the efficiency of the numerical method is an adaptive mesh refinement in the neighborhood of singular points, but if it is necessary to solve a large number of such problems, e.g., in the solution of the synthesis problem, this approach leads to significant computational cost.

Another approach consists in the use of a priori information about the solution behavior, which is introduced in the numerical algorithm. The finite-element method [16, 17] with singular elements that describe the singularity can be cited as an example [18]. However, the addition of new elements to the base can lead to the deterioration of the properties of its Gramian matrix [16] and efficiency reduction of the method.

The numerical algorithm proposed in this paper is based on the construction of the explicit form of the solution singularity in the neighborhood of metal-dielectric corners and its further use in the finite-element method. However, in order to overcome the problems described above, additional finite elements are not added, but some of the basis functions inside and at the boundary of small corner neighborhoods are replaced. Simultaneously, the problem is solved already outside these neighborhoods on a fairly coarse mesh. The use of the additional information about the solution behavior makes the algorithm fast and efficient; this makes it possible to accurately locate the singular part of the solution gradient.

1. FORMULATION OF THE PROBLEM

Let us consider the domain $\Omega = \bigcup_{i=1}^{3} \Omega_i$ with a metal boundary filled with dielectrics with different dielectric constants in each of the subdomains Ω_i (Fig. 1). In Ω, there are corner points of different types: a metal corner (*M*), internal dielectric corners (*D*), and metal-dielectric corners (*C*).

The distribution of the electric potential u is described by the Poisson equation with Dirichlet boundary conditions and conjugation conditions on the interface between dielectrics:

$$\operatorname{div}(\varepsilon_i \operatorname{grad} u) = -f(M), \quad M \in \Omega_i, \quad i = 1\dots3 \quad (1)$$

$$u|_{\partial\Omega} = 0, \quad (2)$$

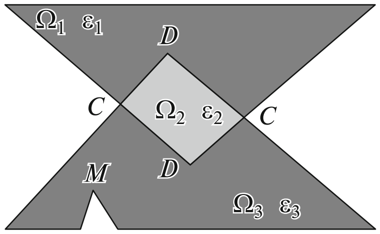

Fig. 1. A domain containing metal (M), dielectric (D), and metal–dielectric (C) corners.

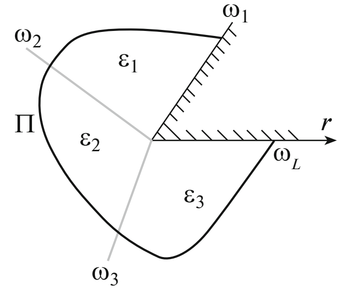

Fig. 2. The geometry of the auxiliary problem for the metal–dielectric corner.

$$u_i\big|_{\partial\Omega_{ij}} = u_j\big|_{\partial\Omega_{ij}}, \tag{3}$$

$$\varepsilon_i \frac{\partial u_i}{\partial n}\bigg|_{\partial\Omega_{ij}} = \varepsilon_j \frac{\partial u_j}{\partial n}\bigg|_{\partial\Omega_{ij}}, \tag{4}$$

where $\varepsilon_i = \text{const}$, $M \in \Omega_i$, $\partial\Omega$ is the total outer metal boundary, $\partial\Omega_{ij}$ is the boundary between the subdomains Ω_i and Ω_j, and $f(M)$ is a sufficiently smooth function expressed in terms of the charge density ρ: $f(M) = 4\pi\rho(M)$.

For the numerical solution of the problem, let us obtain the solution that is asymptotic with respect to smoothness in the neighborhood of all these types of corners.

2.1. CONSTRUCTION OF AN ASYMPTOTIC REPRESENTATION OF THE SOLUTION IN THE NEIGHBORHOOD OF THE METAL-DIELECTRIC CORNER

Let us construct the asymptotic of the solution of problem (1)–(4) in a sufficiently small neighborhood of a Π metal-dielectric corner with the point M_0 (Fig. 2).

Let the ray L emerge from the corner point. We number them so that the first and Lth rays correspond to the metal boundaries. Let us introduce the polar coordinate system centered at this corner point and direct the polar axis along the Lth ray. The angles between the rays and the polar axis will be denoted by ω_i, $i = 1\ldots L$, $\Omega_L = 2\pi$.

In the neighborhood of Π, let us expand the right side $f(M)$ of equation (1) in a Taylor series by r.

Consider the partial sum of the series consisting of the $(P + 1)$ term:

$$\overline{f}(r,\varphi) = C_0 + \sum_{n=1}^{P} r^n C_n(\varphi), \tag{5}$$

where $C_0 = \text{const}$. In the neighborhood of Π, let us replace the right-hand side of equation (1) with its approximation (5) and consider the auxiliary problem (it is necessary to find the bounded quotient in the neighborhood of the zero solution of the equation)

$$\begin{gathered}\operatorname{div}(\varepsilon(\varphi)\operatorname{grad} u) = -\overline{f}(M),\\ M \in \{\Pi\backslash M_0 : \omega_i < \varphi < \omega_{i+1}, i = 1\ldots L-1\},\end{gathered} \tag{6}$$

$$\varepsilon(\varphi) = \varepsilon_i = \text{const}, \quad \text{for} \quad \omega_i < \varphi < \omega_{i+1}.$$

Homogeneous Dirichlet boundary conditions are imposed on rays $\varphi = \omega_1$, ω_L

$$u\big|_{\varphi=\omega_1,\varphi=\omega_L} = 0. \tag{7}$$

Conjugation conditions are imposed on the rays corresponding to the interface of different dielectric constants

$$u\big|_{\varphi=\omega_i+0} = u\big|_{\varphi=\omega_i-0}, \tag{8}$$

$$\varepsilon \frac{\partial u}{\partial n}\bigg|_{\varphi=\omega_i+0} = \varepsilon \frac{\partial u}{\partial n}\bigg|_{\varphi=\omega_i-0}. \tag{9}$$

At present, boundary conditions are imposed on the outer boundary of the neighborhood of Π. Later, conditions of solution continuity and its normal derivative will be given on it.

Since problem (6)–(9) is linear and inhomogeneous, let us present its solution in the form of the sum of the particular solution of the inhomogeneous problem and the general solution of the homogeneous problem, each of which satisfies conditions (7)–(9).

2.2. THE SINGULAR PART OF THE SOLUTION. ADDITIVE SELECTION OF THE SINGULARITY IN THE NEIGHBORHOOD OF THE CORNER POINT

In domains strictly inside dielectrics, nontrivial solutions of homogeneous problem (6)–(9) in polar coordinates with a center at the corner point will be sought in the following form

$$u(r,\varphi) = r^{\sigma}\Phi(\varphi). \tag{10}$$

In each of the subdomains, $\varepsilon(\varphi)$ is constant. Thus, after the separation of variables the solution in the sectors is represented as follows:

$$u_i(r,\varphi) = r^{\sigma}[A_i \sin(\sigma\varphi) + B_i \cos(\sigma\varphi)], \quad \omega_i < \varphi < \omega_{i+1}, \quad i = 1 \ldots L-1, \tag{11}$$

where A_i and B_i are constants. After substituting solution (11) in conjugation conditions (8), (9) we obtain a homogeneous system of linear equations with respect to these constants

$$A_1 \sin(\sigma\omega_m) + B_1 \cos(\sigma\omega_m) = 0, \quad A_{L-1} \sin(2\pi\sigma) + B_{L-1} \cos(2\pi\sigma) = 0, \tag{12}$$

$$A_i \sin(\sigma\omega_i) + B_i \cos(\sigma\omega_i) = A_{i+1} \sin(\sigma\omega_i) + B_{i+1} \cos(\sigma\omega_i), \tag{13}$$

$$\varepsilon(\omega_i - 0)(A_i \cos(\sigma\omega_i) - B_i \sin(\sigma\omega_i)) = \varepsilon(\omega_i + 0)(A_{i+1} \cos(\sigma\omega_i) - B_{i+1} \sin(\sigma\omega_i)), \tag{14}$$

where $i = 1, L-1$, and equations (12) correspond to the metal boundary, and (13)–(14) refer to the boundary between two dielectrics.

The characteristic equation for σ obtained from the condition of equality to zero of the determinant of system (12)–(14) defines nontrivial solutions of this system. If there are positive roots of the characteristic equation $\sigma < 1$, then at the corner point the solution gradient will have a singularity that corresponds to an unlimited increase of the electric field from the physical point of view.

The quantity $\sigma_0 = 0$, for which system (12)–(14) has a slightly different form, should be considered separately. In this case, the general solution will be expressed through a corner linear function:

$$u_i(r,\varphi) = A_i\varphi + B_i, \quad \omega_i < \varphi < \omega_{i+1} \quad i = 1 \ldots L-1.$$

However, from conditions (7)–(9) and the fact that $\varepsilon_i > 0$, it follows that there is only a trivial solution.

Let σ_i be the positive roots of the characteristic equation of system (12)–(14). In this case, the solution at the point M_0 will be bounded. Let us denote the corresponding solutions of (12)–(14) taking the possible multiplicity of the root into account as

$$\chi_i(r,\varphi) = r^{\sigma_i}[A_j^{(i)} \sin(\sigma_i\varphi) + B_j^{(i)} \cos(\sigma_i\varphi)], \quad \omega_j < \varphi < \omega_{j+1}, \quad j = 1 \ldots L-1. \tag{15}$$

The singular part of the solution of problem (6)–(9) is presented as a linear combination of functions of the form (15)

$$u_0(r,\varphi) = \sum_i C_i\chi_i(r,\varphi). \tag{16}$$

2.3. THE SOLUTION OF THE INHOMOGENEOUS PROBLEM. THE SOLUTION REPRESENTATION IN THE NEIGHBORHOOD OF A CORNER POINT

Consider inhomogeneous problem (6)–(9). Because of the linearity of equation (6), we obtain the $(P+1)$ subproblem:

$$\operatorname{div}(\varepsilon(M)\operatorname{grad} u_n) = -r^n C_n(\varphi), \quad n = 0 \ldots P, \quad M \in \{(r,\varphi) \in \Pi{:}\, \omega_i < \varphi < \omega_{i+1}, \quad i = 1 \ldots L-1\}, \tag{17}$$

with additional conditions (7)–(9). The solution of original problem (6) is the sum of the solutions of problems (17), (7)–(9)

$$u = \sum_{n=0}^{P} u_n.$$

By substituting the solution of each of the problems (17), (7)–(9) in the form $u_n = r^s\Phi_n(\varphi)$, we will obtain the following equation:

$$s^2 r^{s-2}\Phi_n(\varphi) + r^{s-2}\Phi_n''(\varphi) = -r^n \frac{C_n(\varphi)}{\varepsilon(\varphi)}, \quad \omega_i < \varphi < \omega_{i+1}, \quad i = 1 \ldots L-1,$$

which implies that $s = n + 2$, and the problem for the nth corner part of $\Phi_n(\varphi)$ is obtained:

$$\Phi_n''(\varphi) + (n+2)^2\Phi_n(\varphi) = -\frac{C_n(\varphi)}{\varepsilon(\varphi)}, \quad \omega_i < \varphi < \omega_{i+1}, \quad i = 1 \ldots L-1, \tag{18}$$

$$\Phi_n\big|_{\varphi=\omega_1} = \Phi_n\big|_{\varphi=\omega_L=2\pi} = 0, \tag{19}$$

$$\Phi_n\big|_{\varphi=\omega_i+0} = \Phi_n\big|_{\varphi=\omega_i-0}, \quad \varepsilon\frac{\partial\Phi_n}{\partial\varphi}\bigg|_{\varphi=\omega_i+0} = \varepsilon\frac{\partial\Phi_n}{\partial\varphi}\bigg|_{\varphi=\omega_i-0} \tag{20}$$

The set of problems (18)–(20) is solved numerically.

Thus, the approximate particular solution of the inhomogeneous problem is as follows:

$$\mu(r,\varphi) = \sum_{n=0}^{P} u_n = \sum_{n=0}^{P} r^{n+2}\Phi_n(\varphi). \tag{21}$$

The solution of the original equation (6)–(9) in the neighborhood of the corner Π is represented as follows

$$\bar{u}(r,\varphi) = \mu(r,\varphi) + \sum_i C_i\chi_i(r,\varphi), \tag{22}$$

where the first term describes the smooth part of the solution and the final sum contains a term with a singularity at the corner point.

2.4. AN INTERNAL DIELECTRIC CORNER

Obtaining the solution asymptotics for a dielectric corner lying within the domain Ω generally repeats a similar derivation to that for the metal corner with the exception of some details.

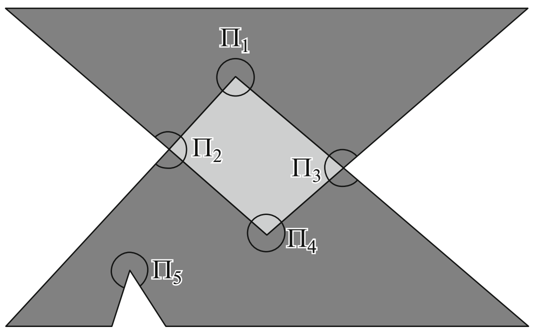

Fig. 3. Introduction of the neighborhoods of corner points.

In this case, the polar coordinate system axis is introduced in the direction of any line of the discontinuity of the dielectric permittivity. Boundary condition (7) is eliminated, and in accordance with the periodicity conditions the coupling condition for $\varphi = 0, 2\pi$ takes the following form:

$$u\big|_{\varphi=0+0} = u\big|_{\varphi=2\pi-0},$$

$$\varepsilon\frac{\partial u}{\partial n}\bigg|_{\varphi=0+0} = \varepsilon\frac{\partial u}{\partial n}\bigg|_{\varphi=2\pi-0}.$$

If $\sigma_0 = 0$, unlike the case with the metal corner, the homogeneous problem has a nontrivial solution, i.e., a constant that should be taken into account in expansion (16).

As a rule, for any dielectric corner configuration, for various σ_i, the fundamental system of solutions (FSS) of system (12)–(14) consists of a single column, but in the case of corners that are multiple to π multiple roots of the characteristic equation can appear a an FSS consisting of several columns.

For a dielectric corner with two dielectrics, the singularity behavior near the corner point was obtained in [9] using the Kondratiev method. Indicators σ_i of the r degree obtained from the characteristic equation of system (13)–(14) correspond to the values obtained in [9].

3. THE NUMERICAL ALGORITHM

Let us turn to the original problem (1)–(4). Let in the domain Ω be S of angles, in which there is a gradient singularity and for each of them the solution representation in the form of (22) was obtained. Let us select neighborhoods of corners Π_s (Fig. 3) in which an approximate solution of problem (1)–(4) is sought as a finite linear combination of functions with degrees r not higher than L:

$$u(r,\varphi) = \mu(r,\varphi) + \sum_{i=0}^{I_{\max}} C_i\chi_i(r,\varphi), \quad C_i \in \mathbb{R}, \qquad (23)$$

where $I_{\max}$ is determined from the condition $\sigma_i < L$. On the introduced boundaries $\partial\Pi_s$, let us impose solution-joining conditions consisting in its continuity and the continuity of its normal derivative.

Let us multiply equation (1) by the test function ν and integrate the resulting expression for the domain $\Omega\backslash\Pi$, where $\Pi = \bigcup_{s=1}^{S}\Pi_s$:

$$\int\limits_{\Omega\backslash\Pi} \operatorname{div}(\varepsilon \operatorname{grad} u)\nu ds = -\int\limits_{\Omega\backslash\Pi} f\nu ds. \qquad (24)$$

By applying Green's formula, we obtain

$$\begin{gathered}-\int\limits_{\Omega\backslash\Pi} \varepsilon \operatorname{grad} u \operatorname{grad} \nu ds + \oint\limits_{\partial\Omega} \varepsilon\frac{\partial u}{\partial n}\nu dl \\ +\sum_{s=1}^{S}\oint\limits_{\partial\Pi} \varepsilon\frac{\partial u}{\partial n}\nu dl = -\int\limits_{\Omega\backslash\Pi} f\nu ds.\end{gathered} \qquad (25)$$

Since boundary condition (2) is the main one, then $\nu|_{\partial\Omega} = 0$, and the first integral of the boundary $\partial\Omega$ vanishes. From the solution joining condition at boundaries $\partial\Pi_s$ of its representation (23) it follows that

$$\begin{gathered}-\int\limits_{\Omega\backslash\Pi} \varepsilon \operatorname{grad} u \operatorname{grad} \nu ds + \sum_{s=1}^{S}\oint\limits_{\partial\Pi} \varepsilon\frac{\partial \mu^{(s)}}{\partial n}\nu dl \\ +\sum_{s=1}^{S}\sum_{i=0}^{I_{\max}} C_i^{(s)}\oint\limits_{\partial\Pi_s} \varepsilon\frac{\partial \chi_i^{(s)}}{\partial n}\nu dl = -\int\limits_{\Omega\backslash\Pi} f\nu ds.\end{gathered} \qquad (26)$$

Thus, the original problem solved in the domain Ω is reduced to the solution of (26) in the domain $\Omega\backslash\Pi$.

The approximate solution of (26) is sought by the finite-element method. Let us denote the nodes of meshes in the domain $\Omega\backslash\Pi$ derived by triangulation through $\{\omega_i\}_{i=1\ldots N}$ (Fig. 4).

Lagrangian elements of the first order $\{\psi_i(x, y)\}_{i=1\ldots N}$ are taken as the basis. Unknown coefficients $C_i^{(s)}$ are added to the column of unknown values of the coefficients in the solution expansion in the finite-element method basis, which are the values of the solution at the mesh nodes. The joining condition of the normal derivative at the boundary $\partial\Pi$ is already taken into account in the derivation of (26), so that during its solution it is only necessary to satisfy the condition of continuity, which is the main one.

The condition of solution continuity at the boundaries $\partial\Pi_s$ is implemented as follows. K_s of nodes $\{\tilde{\omega}_i\}_{i=1\ldots K_s}$ is placed on the boundary $\partial\Pi_s$. Since the problem solution in the subdomain Π_s is defined by formula (23), instead of the standard Lagrangian elements $\{\psi_i(x, y)\}_{i=1\ldots N}$ with vertices $\{\tilde{\omega}_i\}_{i=1\ldots K}$, their linear combinations $\tilde{\psi}_i^{(s)}(x, y)$ are taken at the boundary:

$$\tilde{\psi}_i^{(s)}(x,y) = \sum_{j=1}^{K} A_{ij}^{(s)}\psi_j(x,y), \quad i = 0\ldots I_{\max}, \qquad (27)$$

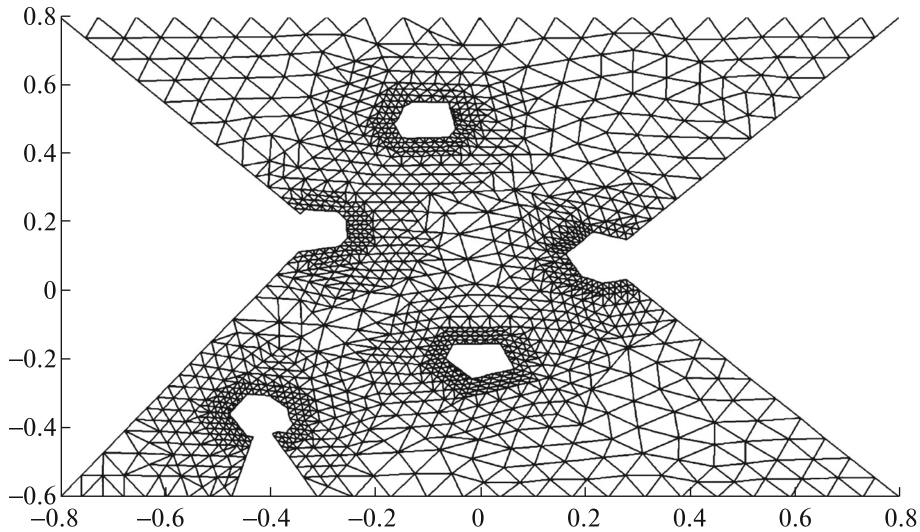

Fig. 4. The mesh in the domain $\Omega\backslash\pi$.

where the matrix with constant coefficients $\{A_{ij}^{(s)}\}$ is determined from the equality condition at the boundary $\partial\Pi_s$ of functions $\chi_i^{(s)}(r,\varphi)$ and $\tilde{\psi}_i^{(s)}(x,y)$. The unknown factors before the introduced functions are taken equal to $C_i^{(s)}$. In addition, the function $\tilde{\mu}^{(s)}(x,y)$ is constructed of standard finite elements at the boundary:

$$\tilde{\mu}^{(s)}(x,y) = \sum_{j=1}^{K} B_j^{(s)}\psi_j(x,y), \tag{28}$$

where the coefficients $B_j^{(s)}$ are determined from the condition of its equality to the smooth part of the solution of $\mu^{(s)}(r,\varphi)$ at the boundary $\partial\Pi_s$. Thus, in the finite-element method, the solution in the domain $\Omega\backslash\Pi$ is sought as an expansion

$$\hat{u}(x,y) = \sum_{s=1}^{S}\left(\tilde{\mu}^{(s)}(x,y) + \sum_{i=0}^{I_{\max}} C_i^{(s)}\tilde{\psi}_i^{(s)}(x,y)\right) + \sum_{j=1}^{N-K} A_j\psi_j(x,y), \tag{29}$$

where the latter sum includes $N - K = N - \sum_{s=1}^{S} K_s$ of original basis functions with vertices strictly inside the domain $\mathbf{\Omega\backslash\Pi}$. Since at the boundaries $\partial\Pi_s$ all $\psi_j(x, y) = 0$, the continuity condition on $\partial\Pi_s$ is fulfilled automatically for the constructed solution based on finite elements.

Retaining the old notation, let us combine two families of basis functions $\{\tilde{\psi}_i^{(s)}(x,y)\}_{i-0\ldots I_{\max}, s=1\ldots S}$ and $\{\psi_j(x, y)\}_{j = 1\ldots N - K}$ in $\{\psi_j(x,y)\}_{i=1\ldots\tilde{N}}$ and the corresponding columns of the unknown coefficients $C_i^{(s)}$ and A_j in $\{C_i\}_{i=0\ldots\tilde{N}}$, where $\tilde{N} = N - K + I_{\max} + 1$ is the total number of basis functions. The approximate solution of (29) can be written as follows

$$\hat{u}(x,y) = \sum_{s=1}^{S}\tilde{\mu}^{(s)}(x,y) + \sum_{i=1}^{\tilde{N}} C_i\psi_i(x,y). \tag{30}$$

Elements of the same base are used as test functions

$$\nu(x,y) = \psi_j(x,y). \tag{31}$$

After substituting (30) and (31) into (26), a system of equations with respect to the unknown coefficients column $\mathbf{C} = \{C_i\}_{i=0}^{\tilde{N}}$ is obtained, which has the following form

$$-\mathbf{K}^{ij}\mathbf{C}_i + \mathbf{Q}^{ij}\mathbf{C}_i = -\mathbf{F}^j + \mathbf{M}^j - \mathbf{G}^j, \tag{32}$$

where

$$\begin{aligned}
\mathbf{K}^{ij} &= \int_{\Omega\backslash\Pi} \varepsilon\,\mathrm{grad}\,\psi_i\,\mathrm{grad}\,\psi_j ds,\\
\mathbf{M}^{j} &= \int_{\Omega\backslash\Pi} \varepsilon\,\mathrm{grad}\,\tilde{\mu}\,\mathrm{grad}\,\psi_j ds,\\
\mathbf{G}^{j} &= \sum_{s=1}^{S}\oint_{\partial\Pi_s} \varepsilon\frac{\partial\mu^{(s)}}{\partial n}\psi_j dl,\\
\mathbf{Q}^{ij} &= \sum_{s=1}^{S}\oint_{\partial\Pi_s} \varepsilon\frac{\partial\psi_i}{\partial n}\psi_j dl,\\
\mathbf{F}^{j} &= \int_{\Omega\backslash\Pi} f\psi_j ds.
\end{aligned} \tag{33}$$

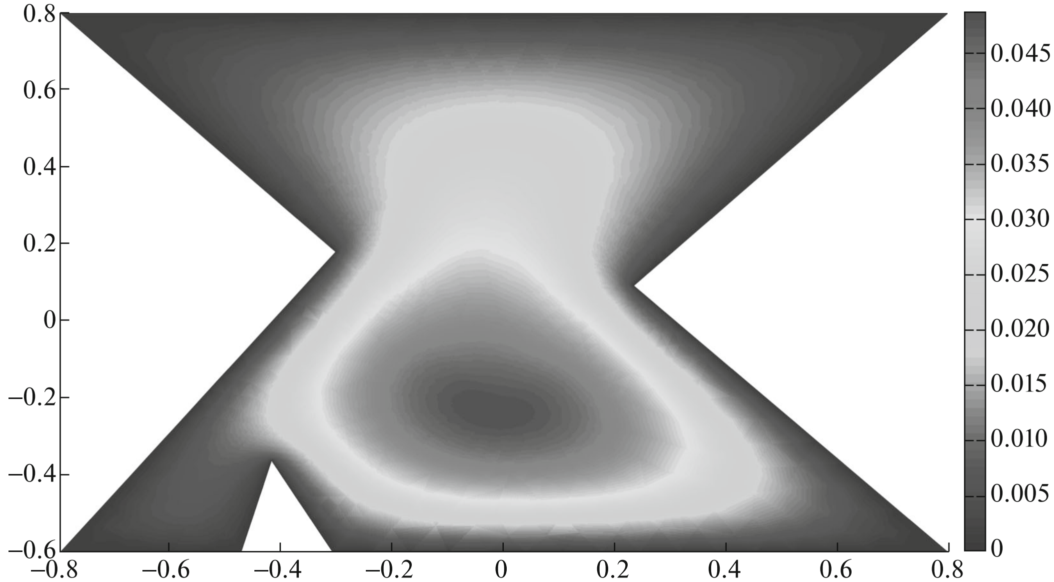

Fig. 5. The solution of (1)–(4), $f(x, y) = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, and $\varepsilon_3 = 3$.

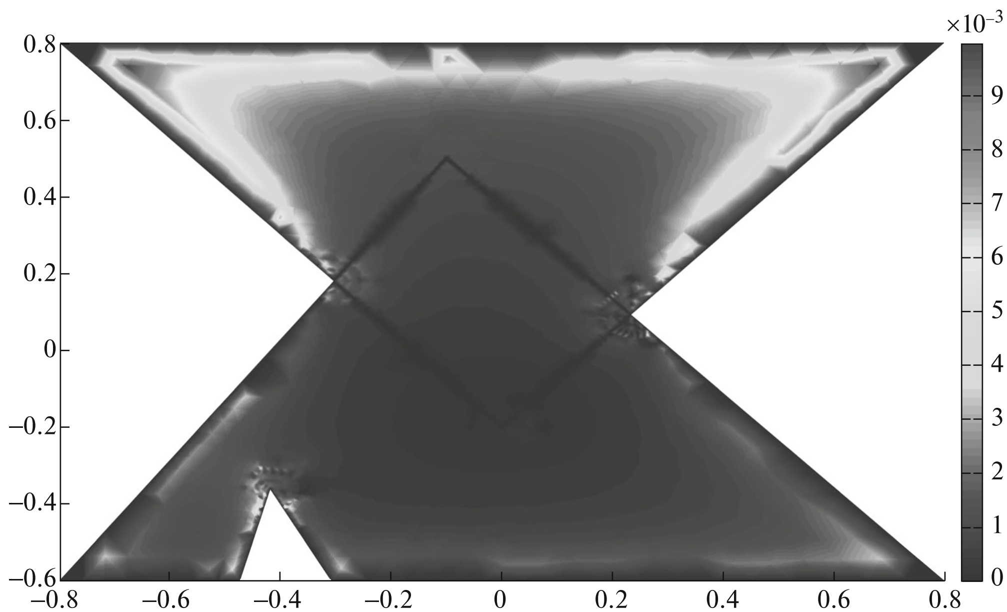

Fig. 6. The relative error of the solution obtained on the coarse mesh taking singularities to the solution into account compared to that on the fine mesh without taking them into account.

Note that when solving the obtained problem (32)–(33) coefficients of the functions that describe the singular part of the solution in each neighborhood Π_s also occur.

4. THE NUMERICAL EXPERIMENT AND VERIFICATION RESULTS

In order to verify the proposed method, problem (1)–(4) was further solved in the entire domain using a standard finite-element method without taking singularities at the corner points on a denser mesh into account.

Figure 5 shows the solution of problem (1)–(4) for $f(x, y) = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, and $\varepsilon_3 = 3$ by the standard finite-element method using a mesh of 11 992 nodes with a strong condensation in the neighborhood of the considered corner points.

Figure 6 shows the relative solution error on a fine mesh without singularity separation compared to the solution obtained by the proposed method on a coarse mesh with 1264 nodes. The discrepancy in the neighborhood of the corners does not exceed 0.4%. When using the same coarse mesh in the conventional finite-element method without the separation of the singular part of the solution, the error in the neighborhoods of corners was up to 20%, which leads to significant errors in the electric field value when calculating the gradient.

6. CONCLUSIONS

In this paper, an algorithm for calculating the electrostatic problem for Poisson's equation in the domain containing metal-dielectric corners was proposed. The algorithm is highly efficient and makes it possible to obtain the distribution of the electric field near the corner points with high accuracy. In addition, this method has good flexibility making it possible to uniformly process metal, dielectric, and metal–dielectric corners with different configurations without significant changes in the solution method.

The method can be used to solve a wide range of electrostatics and electrodynamics problems, such as the problems of analysis and synthesis in domains with complex geometry and that have corner points of different types.

ACKNOWLEDGMENTS

M.I. Svetkin's work was supported by the Russian Foundation for Basic Research, project no. 16-01-00690 A. A.I. Erokhin's work was supported by the Russian Foundation for Basic Research, project no. 16-31-60084 mol_a_dk. A.N. Bogolyubov's and I.E. Mogilevsky's work was supported by the Russian Foundation for Basic Research, project no. 15-01-03524 A.

REFERENCES

1. S. A. Nazarov and B. A. Plamenevskii, *Elliptic Problems in Regions with a Piecewise Smooth Boundary*, (Nauka, Moscow, 1991).
2. V. A. Kondrat'ev and O. A. Oleinik, Russ. Math. Surv. **38** (2), 1 (1983).
3. A. N. Bogolyubov, A. L. Delitsyn, I. E. Mogilevsky, and A. G. Sveshnikov, Zh. Radioelektron., No. 8 (2001). http://jre.cplire.ru.
4. A. N. Bogolyubov, A. L. Delitsyn, I. E. Mogilevsky, and A. G. Sveshnikov, J. Commun. Technol. Electron. **48**, 715 (2003).
5. V. A. Kondrat'ev, Differ. Uravn. **13**, 2026 (1977).
6. S. A. Degtyarev, A. V. Ustinov, and S. N. Khonina, Komp'yut. Opt. **38**, 629 (2014).
7. M. Zokhrabi, Nauchn. Vedomosti Belgorod. Gos. Univ. Ser.: Mat. Fiz. **37** (25), 143 (2014).
8. Yu. A. Tsarin, Radiofiz. Radioastron. **6**, 323 (2001).
9. A. N. Bogolyubov and I. E. Mogilevskii, Comput. Math. Math. Phys. **51**, 2121 (2011).
10. J. M. Jin, *The Finite Element Method in Electromagnetics* (John Wiley & Sons, 2014).
11. O. Davydov and D. T. Oanh, J. Comput. Phys. **230**, 287 (2011).
12. P. Ciarlet and J. He, C. R. Math. **356**, 353 (2003).
13. J. Bedrossian et al., J. Comput. Phys. **229**, 6405 (2010).
14. H. Igarashi and T. Honma, Appl. Math. Model. **20**, 847 (1996).
15. M. Elliotis, G. Georgiou, and C. Xenophontos, Commun. Numer. Methods Eng. **18**, 213 (2002).
16. C. Johnson, *Numerical Solutions of Partial Differential Equations by the Finite Element Method* (Cambridge Univ. Press, 1987).
17. R. H. Gallagher, *Finite Element Analysis: Fundamentals* (Prentice Hall, 1975).
18. G. C. Georgiou, W. W. Schultz, and L. G. Olson, Int. J. Numer. Methods Fluids **10**, 357 (1990).

Translated by O. Pismenov