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THEORETICAL AND MATHEMATICAL PHYSICS

The Quasi-Exactly Solvable Problems for Two Dimensional Quantum Systems[1]

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Abstract– In this paper, we study the quasi-exactly solvable problems for two dimensional quantum systems. By using the Bethe ansatz method, we obtain the general form of the quasi-exactly solvable potential. Then, we present several examples to give the specific forms of quasi-exactly solvable potentials. In the examples, some physical models of quasi-exactly solvable problems are re-exhibited.

Keywords: quasi-exactly solvable, Bethe ansatz method.

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1. INTRODUCTION

In quantum mechanics, the analytical solutions of Schrödinger equation are important, since they contain all the information of the system considered. In practice, the exactly solvable problems are very seldom [1]. Recently, it was found for certain problems, parts of the energy spectrum can be derived analytically, while the rest can not be got, they are the quasi-exactly solvable problems [2–4]. The quasi-exactly solvable problems occupy an intermediate position between the exactly solvable and non-solvable systems, they enlarge the class of exactly solvable problems, and they deserve special attentions.

In the study of quasi-exactly solvable problems, the construction of quasi-exactly solvable potentials becomes crucial [5–11]. For two dimensional quantum systems, the general form of the quasi-exactly solvable potential has not been discussed. In this paper, we study the quasi-exactly solvable potentials for two dimensional quantum problems, and try to give the general form of the quasi-exactly solvable potential.

The organization of the paper is as follows. In section II, we discuss the general form of the quasi-exactly solvable potential; in section III, we give several examples to obtain the specific forms of quasi-exactly solvable potentials; finally, discussions and conclusions are given in section IV.

[1]The article is published in the original.

2. THE GENERAL FORM OF QUASI-EXACTLY SOLVABLE POTENTIAL

For a particle moving in a two dimensional plane, the Schrödinger equation can be written as

$$\left(\frac{P^2}{2M}+V(r)-E\right)\psi=0, \tag{1}$$

where M is the mass of the particle, $V(r)$ stands for the potential, and E is the energy.

We introduce an ansatz of the eigenfunction in polar coordinates

$$\psi(r,\theta)=\frac{u(r)}{\sqrt{r}}\exp(il\theta), \tag{2}$$

where l is the angular momentum quantum number, then Eq. (1) becomes

$$\left[\frac{1}{2M}\left(-\frac{d^2}{dr^2}+\frac{l^2-1/4}{r^2}\right)+V(r)\right]u(r)=Eu(r). \tag{3}$$

If Eq. (3) is quasi-exactly solvable, several exactly solutions can be found. By using the Bethe ansatz method [12–16] we assume the exact solutions has the form of

$$u(r)=\sum_{i=0}^{n}a_i r^i r^{|l|+\frac{1}{2}}\exp\left(-\int\sqrt{M}p_m(r)dr\right), \tag{4}$$

where $p_m(r)$ is a polynomial of degree m. The exact solutions consist of two parts, polynomials and exponential function, which assure the solutions are finite at the limits $r \to 0$ and $r \to \infty$. Then the quasi-exactly solvable potential reads

$$V(r) = \frac{1}{2}\left\{[p_m(r)]^2 + p_{m-1}(r) + \frac{k}{r}\right\}, \tag{5}$$

where $p_{m-1}(r)$ is a polynomials of degree $m-1$, and k is a parameter to be determined. Obviously, the form of quasi-exactly solvable potential depends on the polynomial $p_m(r)$ and the number n, which is the degree of the polynomial in the solution. In the following section we take several examples to give the specific forms of quasi-exactly solvable potentials.

3. EXAMPLES

For simplicity, we choose $p_m(r) = ar$ to give the specific forms of the quasi-exactly solvable potential. The degree of the polynomial $p_m(r)$ is one, $m = 1$, then the degree of polynomial $p_{m-1}(r)$ becomes zero, namely, polynomial $p_{m-1}(r)$ is a constant, $p_{m-1}(r) = c$. From Eq. (5) one gets the quasi-exactly solvable potential

$$V(r) = \frac{1}{2}\left(a^2r^2 + \frac{k}{r}\right), \tag{6}$$

where k is a parameter to be determined, and the constant c is ignored in the potential.

The exact solution of Eq. (3) is

$$u(r) = \sum_{i=0}^{n} a_i r^i r^{|l|+\frac{1}{2}} \exp\left(-\frac{1}{2}\sqrt{M}ar^2\right). \tag{7}$$

It is obvious that the constraint $a > 0$ should be imposed in order to keep the wave function finite.

The parameter k depends on the value of n. For simplicity, we take $n = 1$ and $n = 2$ as examples to give the explicit expressions.

If $n = 1$

If $n = 1$, $u(r)$ becomes

$$u(r) = (a_0 + a_1 r) r^{|l|+\frac{1}{2}} \exp\left(-\int \sqrt{M}ardr\right). \tag{8}$$

Substituting the solution Eq. (8) into the second order differential equation Eq. (3) yields

$$\begin{aligned} \frac{k}{r} - 2E &= \frac{(2|l|+1)a_1}{Ma_0 r} + a\frac{-2|l|-4}{\sqrt{M}} \\ &+ \frac{2a_0 a/\sqrt{M} + (-2|l|-1)a_1^2/(Ma_0)}{a_0 + a_1 r}. \end{aligned} \tag{9}$$

Comparing the two sides of Eq. (9), we obtain

$$\begin{aligned} k &= \frac{(2|l|+1)a_1}{Ma_0}, \\ E &= \frac{a(|l|+2)}{\sqrt{M}}, \\ 0 &= 2a_0 a/\sqrt{M} + (-2|l|-1)a_1^2/(Ma_0). \end{aligned} \tag{10}$$

The parameter k depends on the angular momentum quantum number l and the ratio of the coefficients a_1 to a_0. Solving the third equation in Eq. (10), we get

$$\frac{a_1}{a_0} = \pm\sqrt{\frac{a\sqrt{M}}{|l|+1/2}}; \tag{11}$$

then the parameter k becomes

$$k = \frac{\pm\sqrt{a(4|l|+2)}}{\sqrt[4]{M^3}}. \tag{12}$$

Therefore, for special values of k, a quadratic potential plus Coulomb potential is a quasi-exactly solvable potential, and one of its solution can be obtained analytically.

If $n = 2$

The parameter k will be different from it in Eq. (12), we explore the specific form of it. When $n = 2$, $u(r)$ becomes

$$u(r) = (a_0 + a_1 r + a_2 r^2) r^{|l|+\frac{1}{2}} \exp\left(-\int \sqrt{M}ardr\right). \tag{13}$$

Substituting the solution (13) into Eq. (3), and after simple calculations, we get

$$\begin{aligned} \frac{k}{r} - 2E &= -\frac{4}{\sqrt{M}}a - a\frac{(2|l|+2)}{\sqrt{M}} + \frac{2a_1}{a_0 M}\frac{|l|+\frac{1}{2}}{r} \\ &+ \frac{(-2|l|-1)(a_1 + a_2 r)a_1/(a_0 M) - 2a_1 ar/\sqrt{M} + (4|l|+4)a_2/M + 4a(a_0 + a_1 r)/\sqrt{M}}{a_0 + a_1 r + a_2 r^2}. \end{aligned} \tag{14}$$

Comparing the two sides of Eq. (14), we get the expressions of the energy and parameter k

$$\begin{aligned} E &= a\frac{(|l|+3)}{\sqrt{M}}, \\ k &= (2|l|+1)\frac{a_1}{a_0 M}. \end{aligned} \tag{15}$$

Obviously, the parameter k depends on the ratio of a_1 to a_0. The coefficients a_i in $u(r)$ can be obtained from the following equations

$$\begin{aligned} (-1|l|-1)\frac{a_1^2}{a_0 M} + (4|l|+4)\frac{a_2}{M} + \frac{4}{\sqrt{M}}aa_0 &= 0, \\ (-2|l|-1)\frac{a_1 a_2}{a_0 M} - \frac{2a_1 a}{\sqrt{M}} + \frac{4}{\sqrt{M}}aa_1 &= 0. \end{aligned} \tag{16}$$

Solving the equation, one can easily get

$$\frac{a_1}{a_0} = \pm\frac{\sqrt[4]{M}\sqrt{a(4|l|+3)}}{|l|+1/2},\quad \frac{a_2}{a_0} = \frac{a\sqrt{M}}{|l|+1/2}. \tag{17}$$

Then, the parameter k becomes

$$k = \pm\frac{\sqrt[2]{a(4|l|+3)}}{\sqrt[4]{M^3}}. \tag{18}$$

Therefore, if the parameter a in the potential Eq. (6) is fixed, only for special parameter k, the second order differential equation is quasi-exactly solvable. Or equivalently saying, if k is fixed, only for special values of a are allowed.

As one knows, k/r can be seen as the effect of Coulomb field, if we choose $a = \sqrt{M}\omega$, ω is the frequency of the harmonic-oscillator potential, then, Eq. (6) can describe the motion of a hydrogen-like atom in an electronic harmonic-oscillator potential field, hence, it is a physical model of quasi-exactly solvable problem. The recent studies shows that the motion of a charged particle in Coulomb and magnetic fields [17–22] and the relative motion of two charged particles in an external oscillator potential [23–25] are all the physical models of quasi-exactly solvable problems, the potentials of them can reduce to the basic equation Eq. (6).

4. CONCLUSION

In this paper, the quasi-exactly solvable problems for two dimensional quantum systems are studied. By using the Bethe ansatz method, the general form of the quasi-exactly solvable potential is obtained, and several examples are presented to give the specific forms of them. It shows for special parameters a quadratic potential plus Coulomb potential can be quasi-exactly solvable potential, exact solutions of the Schrödinger equations can be obtained. The physical models of this quasi-exactly solvable problem are the motion of electrons in external electric and magnetic fields. Besides the potential constructed in the examples, many other forms can be constructed by varying the form of the polynomial $p_m(r)$. For example, if $p_m(r)$ is chosen as $p_m(r) = ar + c$, then a quadratic potential plus Cornell potential $br + k/r$, $V(r) = (a^2r^2 + br + k/r)/2$, can be quasi-exactly solvable potential for special parameters k.

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REFERENCES

1. J. Zeng, *Quantum Mechanics* (Science Press, Beijing, 2000).
2. M. A. Shifman, in *Contemporary Mathematics*, Vol. 160: *Lie Algebras, Cohomology, and New Applications to Quantum Mechanics*, Ed. by N. Kamran and P. J. Olver (American Mathematical Society, 1994), p. 237.
3. M. A. Shifman and A. V. Turbiner, Commun. Math. Phys. **126**, 347 (1989).
4. A. V. Turbiner, Commun. Math. Phys. **118**, 467 (1988).
5. M. A. Shifman, *ITEP Lectures on Particle Physics and Field Theory* (World Scientific, Singapore, 1999).
6. A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (IOP, London, 1994).
7. A. Gangopadhyaya, A. Khare, and U. P. Sukhatme, Phys. Lett. A **208**, 261 (1995).
8. M. Taut, J. Phys. A **27**, 1045 (1994).
9. D. Agboola and Y.-Z. Zhang, Ann. Phys. **330**, 246 (2013).
10. D. Agboola and Y.-Z. Zhang, Ann. Phys. **327**, 2275 (2012).
11. C.-L. Ho, Ann. Phys. **326**, 797 (2011).
12. H. Bethe, Z. Phys. **71**, 205 (1931).
13. P. P. Kulish and E. K. Sklyanin, Phys. Lett. A **70**, 461 (1979).
14. V. E. Korepin, A. G. Izergin, and N. M. Bogoliubov, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).
15. Y.-Z. Zhang, J. Phys. A **45**, 065206 (2012).
16. D. Agboola and Y.-Z. Zhang, Ann. Phys. **327**, 2275 (2012).
17. M. Taut, J. Phys. A **28**, 2081 (1995).
18. M. Taut, J. Phys. A **32**, 5509 (1999).
19. V. M. Villalba and R. Pino, Phys. Lett. A **49**, 238 (1998).
20. C.-L. Ho and V. R. Khalilov, Phys. Rev. A **61**, 032104 (2000).
21. C.-M. Chiang and C.-L. Ho, Phys. Rev. A **63**, 062105 (2001).
22. C.-L. Ho and P. Roy, Ann. Phys. **312**, 161 (2004).
23. M. Taut, Phys. Rev. A **48**, 3561 (1993).
24. L. Liu and Q. Hao, Theor. Math. Phys. **183**, 730 (2015).
25. D. Yu. Cherkasov, Moscow Univ. Phys. Bull. **67**, 335 (2012).