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THEORETICAL AND MATHEMATICAL PHYSICS

Superconductivity in Chiral-Asymmetric Matter within the (2 + 1)-Dimensional Four-Fermion Model

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Abstract—The phase structure of chiral-asymmetric matter has been studied within the (2 + 1)-dimensional quantum-field theory with the fermion–antifermion and fermion–fermion (or superconducting) channels of four-fermion interaction. For this purpose, the model takes both the chemical potential of the number of particles μ and the chiral chemical potential μ_5 conjugated to the difference between the numbers of right and left fermions into account. A series of phase diagrams was plotted for different chemical potentials. It is shown that the chemical potential μ promotes the appearance of a superconducting phase, while an increase in the chemical potential μ_5 suppresses the effect of the chemical potential μ on a system. The results of this study may be of interest for high-energy physics, condensed matter physics and, in particular, graphene physics.

Keywords: four-fermion model, phase transitions, chiral matter, superconductivity.

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INTRODUCTION

It is well known that the relativistic quantum-field models with four-fermion interactions serve as an efficient theory for the description of real physical effects. As an example, the model with four-fermion interaction in a (3 + 1)-dimensional time-space [1–3], which is also known as the Nambu–Jona-Lasinio (NJL) model [4], is often used in studies on meson spectroscopy and neutron star physics and experiments of heavy-ion accelerators. Let us also note that four-fermion models attract special interest in recent years as a theory that describes phenomena in condensed matter physics. In fact, the physics of a (quasi)-one-dimensional Peierls dielectric (polyacetylene is the best known representative) is describable with the (1 + 1)-dimensional four-fermion Gross–Neveu (GN) model [5–11]. It is important to point out that the (2 + 1)-dimensional GN model variant is applied for the description of high-temperature conductivity and some other effects in materials with a planetary crystal structure, including graphene (a planar monoatomic layer of carbon atoms) [12–18]. It is important that some physical effects also exist that were first discovered within the NJL and GN models. In particular, this is the dynamic fermion mass generation effect, which was first described by Y. Nambu and G. Jone-Lasinio [4] as a distortion of continuous chiral symmetry within a four-fermion model. This effect was later successfully applied for the description of a low-energy spectrum of mesons in quantum chromodynamics (QCD) [1].

It is also convenient to describe the dynamic symmetry distortion effect within low-dimensional ($D = 1 + 1$ and $D = 2 + 1$) models of four-fermion interactions [5, 19, 20]. It is worth emphasizing that a model is renormalizable in a (1 + 1)-dimensional time-space and renormalizable within the expansion as a series in $1/N$, where N is the number of fermion fields, in a (2 + 1)-dimensional case [20].

It is important to note that matter with a non-zero chiral chemical potential μ_5, i.e., chiral-asymmetric (or simply chiral) matter, also has attracted special interest in recent years. The densities of left- (n_L) and right-handed (n_R) fermions are different in chiral media. Chiral matter may be formed as a result of collisions between heavy ions, in compact stars, condensed media, etc. [21–25]. Hence, it is important to have a theoretical model of chiral matter that may be formed under such diverse conditions.

In this paper we describe chiral-asymmetric matter using the GN model in a (2 + 1)-dimensional space with consideration for two channels, chiral and superconducting, of four-fermion interaction between particles. We attempted to avoid the loss of generality in the formulation and study of this model, which is suitable for both the description of solid and high-energy physics. However, in contrast to our previous work [26], the model we consider here is invariant to discrete symmetry instead of continuous γ_5-symmetry. In this work, we also consider only one of two possible γ_5-matrix representations, which is most often used in condensed-matter physics (we will return to this question in the Conclusions).

1. THE MODEL AND ITS EFFECTIVE POTENTIAL

Our study is based on the (2 + 1)-dimensional Gross–Neveu model with massless fermions belonging to the fundamental multiplet of the flavor group $SO(N)$. Its Lagrangian describes the interaction in both the fermion–fermion (superconducting) and fermion–antifermion (chiral) channels as

$$L = \sum_{k=1}^{N} \bar{\psi}_k[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5]\psi_k + \frac{G_1}{N}\left(\sum_{k=1}^{N}\bar{\psi}_k\psi_k\right)^2 + \frac{G_2}{N}\left(\sum_{k=1}^{N}\psi_k^{\mathrm{T}}C\psi_k\right)\left(\sum_{k=1}^{N}\bar{\psi}_kC\bar{\psi}_k^{\mathrm{T}}\right), \quad (1)$$

where μ is the chemical potential of the number of fermions (conjugated to the density of fermions) and μ_5 is the chiral chemical potential (conjugated to the chiral density $n_5 = n_{\mathrm{R}} - n_{\mathrm{L}}$). As mentioned above, all the fermion fields ψ_k ($k = 1,\ldots, N$) represent the fundamental multiplet of the flavor group $SO(N)$. Hereinafter, summation over repeated indices $k = 1,\ldots, N$ and ν = 0, 1, 2 is implied. Each field ψ_k is a four-component (reducible) Dirac spinor (the superscript T implies transposition). γ^ν (ν = 0, 1, 2, 5) are matrices in a four-dimensional spinor space. Their form in this reducible representation of the $SO(2,1)$ group can easily be found in the literature (see, e.g., [16, Appendix A]). In particular, the γ_5 matrix $\equiv i\begin{pmatrix}0, & -I\\ I, & 0\end{pmatrix}$ where I is a 2 × 2 unit matrix, is used as γ_5 in our work. Moreover, the notation $C \equiv \gamma^2$ is used in Eq. (1) for the charge conjugation matrix.

It is obvious that the Lagrangian L is invariant with respect to the transformations of the flavor group $SO(N)$, which was introduced to perform calculations using the nonperturbative method of expansion in $1/N$ in the case of great N. It is physically more interesting that the model is invariant with respect to the discrete chiral transformation $\psi_k \to \gamma_5\psi_k$ and the continuous group of transformations $U(1)$ $\psi_k \to \exp(i\alpha)\psi_k$ responsible for the retention of an electrical charge.

The linearized version of Lagrangian (1) with the auxiliary boson fields $\sigma(x)$, $\Delta(x)$, and $\Delta^*(x)$ has the following form:

$$\mathcal{L} = \bar{\psi}_k[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \sigma]\psi_k - \frac{N\sigma^2}{4G_1} - \frac{N}{4G_2}\Delta^*\Delta - \frac{\Delta^*}{2}[\psi_k^{\mathrm{T}}C\psi_k] - \frac{\Delta^*}{2}[\bar{\psi}_kC\bar{\psi}_k^{\mathrm{T}}]. \quad (2)$$

It is probable that Lagrangians (1) and (2) are equivalent, as may be ascertained using the Euler–Lagrange motion equations for boson fields in the following form:

$$\sigma(x) = -2\frac{G_1}{N}(\bar{\psi}_k\psi_k), \quad \Delta(x) = -2\frac{G_2}{N}(\psi_k^{\mathrm{T}}C\psi_k), \quad \Delta^*(x) = -2\frac{G_2}{N}(\bar{\psi}_kC\bar{\psi}_k^{\mathrm{T}}). \quad (3)$$

From Eqs. (3) it can clearly be seen that the neutral field $\sigma(x)$ is a Hermitean value, i.e., $\sigma(x)^\dagger = \sigma(x)$, where † means Hermitean conjugation. It also follows from Eqs. (3) that the charged difermion fields $\Delta(x)$ and $\Delta^*(x)$ satisfy the relationships $\Delta(x)^\dagger = \Delta^*(x)$ and $\Delta(x)^{*\dagger} = \Delta(x)$. It is obvious that all fields (3) are singlets with respect to the transformations of the $SO(N)$ flavor group. Moreover, the fields $\sigma(x)$, $\Delta(x)$, and $\Delta^*(x)$ are even values, i.e., scalars with respect to the evenness transformation P

$$P:\ \psi_k(t,x,y) \to \gamma^5\gamma^1\psi_k(t,-x,y), \quad k = 1,\ldots,N. \quad (4)$$

If the difermion field $\Delta(x)$ has a non-zero average value with respect to the ground state ($\langle\Delta(x)\rangle \neq 0$), the Abelian symmetry group responsible for the retention of a charge is spontaneously destroyed and a phase of superconductivity appears in the model. In the case $\langle\sigma(x)\rangle \neq 0$, chiral symmetry proves to be spontaneously destroyed and, as a consequence, fermions acquire a mass.

Let us study the phase structure of model (1) using equivalent Lagrangian (2). In the approximation of $N \to \infty$, the effective action $\mathcal{S}_{\mathrm{eff}}(\sigma, \Delta, \Delta^*)$ has the following form [20]:

$$\exp(i\mathcal{S}_{\mathrm{eff}}(\sigma,\Delta,\Delta^*)) = \int\prod_{l=1}^{N}[d\bar{\psi}_l][d\psi_l]\exp\left(i\int\mathcal{L}d^3x\right), \quad \text{where} \quad (5)$$

$$\mathcal{S}_{\mathrm{eff}}(\sigma,\Delta,\Delta^*) = -\int d^3x\left[\frac{N}{4G_1}\sigma^2(x) + \frac{N}{4G_2}\Delta(x)\Delta^*(x)\right] + \tilde{\mathcal{S}}_{\mathrm{eff}}. \quad (6)$$

The fermion contribution into the effective action, i.e., the term $\tilde{\mathcal{S}}_{\mathrm{eff}}$, looks as follows:

$$\exp(i\tilde{\mathcal{S}}_{\mathrm{eff}}) = \int\prod_{l=1}^{N}[d\psi_l][d\psi_l] \times \exp\left\{i\int\left[\psi_k(\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \sigma)\psi_k - \frac{\Delta^*}{2}(\psi_k^{\mathrm{T}}C\psi_k) - \frac{\Delta}{2}(\bar{\psi}_kC\bar{\psi}_k^{\mathrm{T}})\right]d^3x\right\}. \quad (7)$$

The ground state of the compound boson fields $\langle\sigma(x)\rangle, \langle\Delta(x)\rangle$, and $\langle\Delta^*(x)\rangle$ is determined from the effective action steadiness condition

$$\frac{\delta\mathcal{S}_{\mathrm{eff}}}{\delta\sigma(x)} = 0, \quad \frac{\delta\mathcal{S}_{\mathrm{eff}}}{\delta\Delta(x)} = 0, \quad \frac{\delta\mathcal{S}_{\mathrm{eff}}}{\delta\Delta^*(x)} = 0. \quad (8)$$

Further, we shall assume that the ground state of fields does not depend on the coordinates, i.e., $\langle\sigma(x)\rangle \equiv M$, $\langle\Delta(x)\rangle \equiv \Delta$, and $\langle\Delta^*(x)\rangle \equiv \Delta^*$, where M, Δ, and Δ^* are constant values. In actual practice, they represent the coordinates of the global minimum of the thermodynamic potential (TDP) $\Omega(M, \Delta, \Delta^*)$. In the leading order by $1/N$, the thermodynamic potential looks as follows:

$$\int d^3x\Omega(M,\Delta,\Delta^*) = -\frac{1}{N} \times \mathcal{S}_{\text{eff}}\{\sigma(x),\Delta(x),\Delta^*(x)\}\big|_{\sigma(x)=M,\Delta(x)=\Delta,\ldots}. \quad (9)$$

Hence, to determine the ground states of boson fields, it is sufficient for us to find the global TDP minimum point $\Omega(M, \Delta, \Delta^*)$ using the following set of equations:

$$\frac{\partial\Omega(M,\Delta,\Delta^*)}{\partial M} = 0, \quad \frac{\partial\Omega(M,\Delta,\Delta^*)}{\partial\Delta} = 0, \quad \frac{\partial\Omega(M,\Delta,\Delta^*)}{\partial\Delta^*} = 0. \quad (10)$$

Taking Eqs. (5), (6), and (9) into account, we obtain the following TDP calculation formula:

$$\int d^3x\Omega(M,\Delta,\Delta^*) = \int d^3x\left(\frac{M^2}{4G_1} + \frac{\Delta\Delta^*}{4G_2}\right) + \frac{i}{N}\ln\left(\int\prod_{l=1}^{N}[d\bar{\psi}_l][d\psi_l] \times \exp\left(i\int d^3x\left[\bar{\psi}_k D\psi_k - \frac{\Delta^*}{2}(\psi_k^{\mathrm{T}}C\psi_k) - \frac{\Delta}{2}(\bar{\psi}_k C\bar{\psi}_k^{\mathrm{T}})\right]\right)\right), \quad (11)$$

where $D = \gamma^\rho i\partial_\rho + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - M$. For further convenience, we shall assume without loss of generality that Δ and Δ^* are reals[1] such that $\Delta = \Delta^* \equiv \Delta$, where Δ is a real. After taking the integral over fermion fields[2], the non-renormalizable TDP looks as follows:

$$\Omega^{\text{un}}(M,\Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} + \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\ln[(p_0^2 - E_{1+}^2) \times (p_0^2 - E_{1-}^2)(p_0^2 - E_{2+}^2)(p_0^2 - E_{2-}^2)], \quad (12)$$

where

$$E_{1\pm}^2 = (\mu_5 + |\mathbf{p}|)^2 + \mu^2 + M^2 + \Delta^2 \pm 2\sqrt{M^2(\mu^2 + \Delta^2) + \mu^2(\mu_5 + |\mathbf{p}|)},$$
$$E_{2\pm}^2 = (\mu_5 - |\mathbf{p}|)^2 + \mu^2 + M^2 + \Delta^2 \pm 2\sqrt{M^2(\mu^2 + \Delta^2) + \mu^2(\mu_5 - |\mathbf{p}|)}. \quad (13)$$

[1] The phases of complex Δ and Δ^* may be excluded via the corresponding transformation of fermion fields in continual integral (11).
[2] The continual integral calculation method was described in detail in [26] and the textbook [27].

From Eqs. (13) it can clearly be seen that it is further possible to consider only non-negative values of the dynamic variables ($M \geq 0$ and $\Delta \geq 0$) and the chemical potentials ($\mu \geq 0$ and $\mu_5 \geq 0$) without loss of generality. Integrating Eq. (12) over the variable p_0, we obtain

$$\Omega^{\text{un}}(M,\Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \frac{1}{2}\int\frac{d^2p}{(2\pi)^2}(E_{1+} + E_{1-} + E_{2+} + E_{2-}). \quad (14)$$

It is obvious that

$$E_{1+} + E_{1-} + E_{2+} + E_{2-} = 4|\mathbf{p}| + \frac{2(M^2 + \Delta^2)}{|\mathbf{p}|} + \mathcal{O}(1/|\mathbf{p}|^3). \quad (15)$$

Hence, Eq. (14) is an UV-divergent value, which must be renormalized. Since expansion (15) does not depend on μ and μ_5, it is possible to decrease the subintegral expression of Eq. (14) by subtracting its value at $\mu = \mu_5 = 0$, i.e.,

$$(E_{1+} + E_{1-} + E_{2+} + E_{2-})\big|_{\mu=\mu_5=0} = \sqrt{|\mathbf{p}|^2 + (M + \Delta)^2} + \sqrt{|\mathbf{p}|^2 + (M - \Delta)^2} \quad (16)$$

and thereby to obtain a convergent integral. For the transformation to be identical, it is necessary to add the corresponding divergent integral. As a result, the thermodynamic potential takes the form

$$\Omega^{\text{un}}(M,\Delta) = V^{\text{un}}(M,\Delta) - A(M,\Delta), \quad \text{where} \quad (17)$$

$$V^{un}(M,\Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \int\frac{d^2p}{(2\pi)^2}\left(\sqrt{|\mathbf{p}|^2 + (M + \Delta)^2} + \sqrt{|\mathbf{p}|^2 + (M - \Delta)^2}\right), \quad (18)$$

$$A(M,\Delta) = \frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\Big[E_{1+} + E_{1-} + E_{2+} + E_{2-} - 2\sqrt{|\mathbf{p}|^2 + (M + \Delta)^2} - 2\sqrt{|\mathbf{p}|^2 + (M - \Delta)^2}\Big]. \quad (19)$$

Let us emphasize once again that the integral $A(M, \Delta)$ converges, and all divergence is contained in the summand $V^{\text{un}}(M, \Delta)$. To renormalize $V^{\text{un}}(M, \Delta)$, it is necessary to regularize it at first, confining the integration in Eq. (18) within the regions $|p_1| < \Lambda$ and $|p_2| < \Lambda$. Further, it is necessary to require the existence of a dependence between the bare coupling constants G_1 and G_2 and Λ such that both Eq. (18) and entire TPD (17) in the limit $\Lambda \to \infty$ will become finite values, which we denote as $V^{\text{ren}}(M, \Delta)$ and $\Omega^{\text{ren}}(M, \Delta)$, respectively. Without getting into detail,[3] let us cite some

[3] The renormalization procedure was described in detail in [16].

divergence-reducing expressions for the coupling constants:

$$\frac{1}{4G_1} \equiv \frac{1}{4G_1(\Lambda)} = \frac{2\Lambda \ln(1+\sqrt{2})}{\pi^2} + \frac{1}{2\pi g_1},$$
$$\frac{1}{4G_2} \equiv \frac{1}{4G_2(\Lambda)} = \frac{2\Lambda \ln(1+\sqrt{2})}{\pi^2} + \frac{1}{2\pi g_2}, \quad (20)$$

where $g_{1,\,2}$ are finite-model parameters, which are independent from Λ and have the reciprocal mass dimension. Since the bare coupling constants G_1 and G_2 are invariant with respect to the renormalized group, this property is also inherent in the constants $g_{1,\,2}$. Hence, the resulting expression for the thermodynamic potential is obtained from Eq. (17) by means of the subtraction procedure with allowance for Eq. (20):

$$\Omega^{\text{ren}}(M,\Delta) = V^{\text{ren}}(M,\Delta) - A(M,\Delta), \quad (21)$$

where $V^{\text{ren}}(M,\Delta)$ is the effective model potential in a vacuum, i.e., at $\mu = \mu_5 = 0$:

$$V^{\text{ren}}(M,\Delta) = \frac{M^2}{2\pi g_1} + \frac{\Delta^2}{2\pi g_2} + \frac{(M+\Delta)^3}{6\pi} + \frac{|M-\Delta|^3}{6\pi}. \quad (22)$$

Numerical analysis shows that the global minimum of thermodynamic potential (21) can be located only on the axis M and Δ of the plane (M, Δ). For this reason, to find the global minimum of thermodynamic potential (21), it is sufficient to consider the projections of $\Omega^{\text{ren}}(M, \Delta)$ onto these axes, i.e., compare the minima of the functions $F_1 = \Omega^{\text{ren}}(M, 0)$ and $F_2(\Delta) = \Omega^{\text{ren}}(0, \Delta)$, where

$$F_1(M) = \frac{M^2}{2\pi g_1} + \frac{(\mu_5^2+M^2)^{3/2}}{3\pi} - \frac{\theta(\mu - \sqrt{M^2+\mu_5^2})}{6\pi}[\mu^3 - 3\mu(M^2-\mu_5^2) + 2(\mu_5^2+M^2)^{3/2}]$$
$$- \frac{\theta(\sqrt{M^2+\mu_5^2}-\mu)}{2\pi}\left[\mu_5^2\sqrt{\mu_5^2+M^2} + \mu_5 M^2 \ln\left(\frac{\mu_5+\sqrt{\mu_5^2+M^2}}{M}\right)\right]$$
$$- \frac{\theta(\mu-M)\theta(\sqrt{M^2+\mu_5^2}-\mu)}{2\pi}\left[\mu_5\mu\sqrt{\mu^2-M^2} - \mu_5 M^2 \ln\left(\frac{\mu+\sqrt{\mu^2-M^2}}{M}\right)\right],$$
$$F_2(\Delta) = \frac{\Delta^2}{2\pi g_2} + \frac{1}{2\pi}\sum_{\eta=\pm}\left\{\frac{[(\mu+\eta\mu_5)^2+\Delta^2]^{3/2}}{3} - \frac{(\mu+\eta\mu_5)^2}{2}\sqrt{(\mu+\eta\mu_5)^2+\Delta^2}\right.$$
$$\left. - \frac{(\mu+\eta\mu_5)\Delta^2}{2}\ln\left|\frac{\mu+\eta\mu_5+\sqrt{(\mu+\eta\mu_5)^2+\Delta^2}}{\Delta}\right|\right\}. \quad (23)$$

It should be noted that these expressions coincide with the corresponding equations for the thermodynamic potential from the work [16] at $\mu_5 = 0$.

2. PHASE STRUCTURE OF THE MODEL

To study the phase structure of the model, it is necessary to solve the problem of finding the global minimum point (GMP) for thermodynamic potential (21). As mentioned above, the location of the TDP minimum point may correspond to three different phases, which can appear in the model, i.e., (i) if GMP has the form (0,0), the system is in a symmetric phase; (ii) if the minimum of thermodynamic potential (21) is located at a point $(M \neq 0,0)$, we have a phase with spontaneous chiral invariance distortion; (iii) if GMP has the form $(0, \Delta \neq 0)$, the system is in a superconducting phase.

Within the framework of our model, it is sufficient to compare the minima of the functions $F_1(M)$ and $F_2(\Delta)$ with each other to find the phase structure; this determines the global minimum point of thermodynamic potential (21) in the long run. Below, we have plotted several model (g_1, g_2)-phase diagrams (portraits), which give some information about the TDP global minimum point type depending on the coupling constants g_1 and g_2 at different fixed values of the chemical potentials μ and μ_5.

Let us begin with the vacuum case $\mu = \mu_5 = 0$. The result of searching for the global minimum point of Eq. (22) is shown in the phase portrait of Fig. 1. The regions denoted in Fig. 1 as I, II, and III correspond to a symmetric phase, a phase with chiral symmetry distortion, and a phase of superconductivity.

Further, a series of model (g_1, g_2)-phase diagrams at random fixed non-zero μ_5 and different chemical potential μ, such as $\mu = 0$, $0.2\mu_5$, $0.5\mu_5$, and μ_5 are shown in Figs. 2a–2d, respectively. The dimensionless values $g_1\mu_5$ and $g_2\mu_5$ are plotted along the axes in Fig. 2b–2d. Let us point to some characteristic model properties for this selection of extremal parameters. First, it can be seen that symmetric phase I is absent in

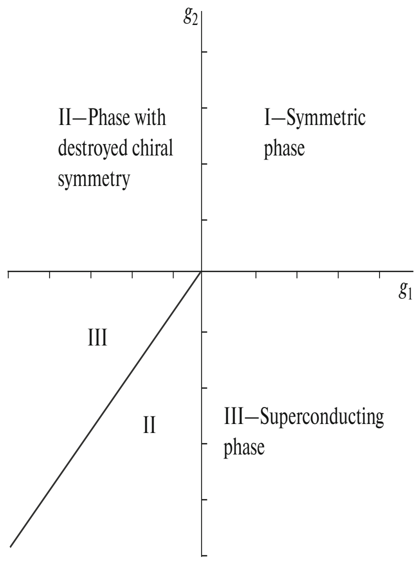

Fig. 1. The g_1, g_2-phase portrait of the model at $\mu = \mu_5 = 0$. At $g_{1,\,2} < 0$, the line l is defined as $l \equiv \{(g_1, g_2)\colon g_1 = g_2\}$.

the phase portrait of Fig. 2a at fixed $\mu_5 \neq 0$ and $\mu = 0$. Second, when comparing the phase diagrams shown in Figs. 2b–2d, it is possible to establish the evolution of the superconducting phase of this model with an increase in the chemical potential μ, namely, the region occupied by superconducting phase III is enlarged with increasing μ.

Let us consider another series of (g_1, g_2)-diagrams corresponding to the same random non-zero value of μ at different also fixed μ_5 (see Fig. 3). In this case, the diagram axes correspond to the other dimensionless parameters $g_1\mu$ and $g_2\mu$. The parameter μ_5 in Figs. 3a–3d is $\mu_5 = 0$, 3μ, 5μ, and 10μ, respectively. It should be noted that the symmetric phase is also absent in this case. A more essential result consists in the fact that a (g_1, g_2)-phase diagram tends to the type shown in Fig. 2a with increasing μ_5 (at fixed μ). In other words, the model (g_1, g_2)-phase portraits at $\mu_5 \neq 0$, $\mu = 0$ and $\mu_5 \to \infty$, $\mu \neq 0$ are similar, i.e., an increase in the chiral chemical potential μ_5 may compensate the effect of the fermion chemical potential μ on the system.

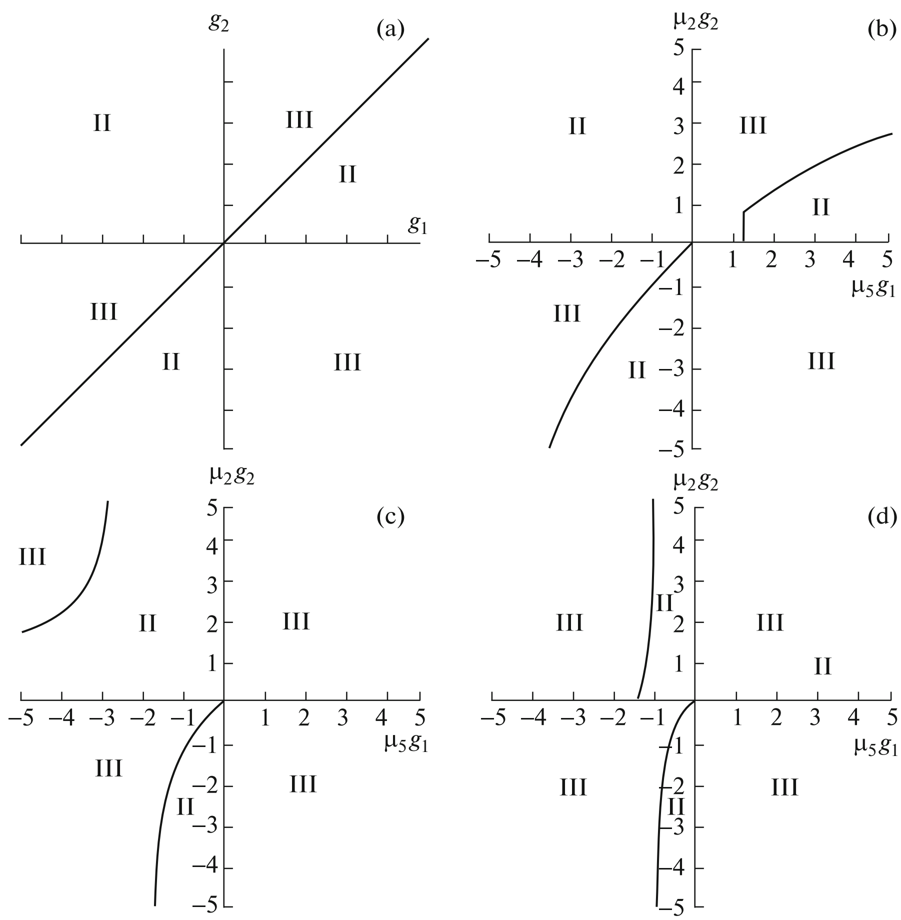

Fig. 2. A series of (g_1, g_2)-phase diagrams of the model at random fixed non-zero μ_5 and different chemical potential μ: $\mu = 0$, $0.2\mu_5$, $0.5\mu_5$, and μ_5, respectively. Phase notations are the same as in Fig. 1.

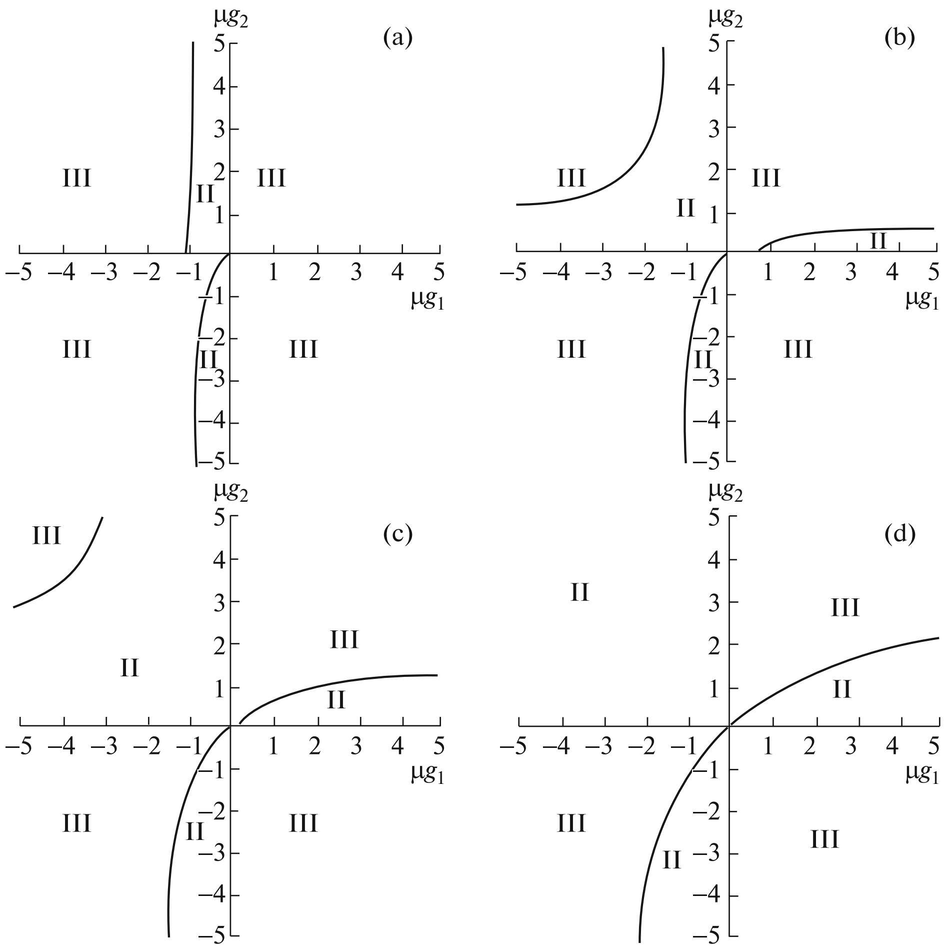

Fig. 3. A series of (g_1, g_2)-phase diagrams of the model at random fixed non-zero μ and different chemical potential μ_5: $\mu_5 = 0$, 3μ, 5μ, and 10μ, respectively. Phase notations are the same as in Fig. 1.

CONCLUSIONS

In this work, we have considered a particular case of interaction between fermions including both the chiral and superconducting channels of interaction. Interactions of such a type may be observed in high-energy physics and condensed-matter physics. The selected lowered space dimension has been dictated by the interest in graphene-like materials and high-temperature superconductors, which have a planar crystal structure and can efficiently be described in the (2 + 1)-dimensional time–space.

The four-fermion interaction model, which is currently applied for the description of a broad spectrum of phenomena, was used in this study. Special attention was focused on the chirality of matter as provided by the introduced chemical potential μ_5. The formation of this type of matter is expected in experiments on the collision between heavy ions and experiments on the doping of graphene-like materials.

The interaction Lagrangian was used as a basis to construct the thermodynamic potential and calculate a series of (g_1, g_2)-phase diagrams at different external parameters. It has been revealed from the behavior of phase diagrams depending on an increase in the chemical potentials that the fermion chemical potential μ induces the formation of the phase of superconductivity within a broader range of the coupling constants. An increase in the chiral chemical potential μ_5 prevents the effect of the fermion chemical potential μ on the system. This result with a corresponding interpretation may be of practical interest for different fields of physics.

Let us now discuss the role played by the form of the chiral transformation γ^5 matrix in the phase transitions in the (2 + 1)-dimensional GN models. Let us remember that just the reducible four-dimensional representation of the $SO(2,1)$ group for Dirac spinors enables the introduction of chiral transformations into such models (the non-reducible representation of the $SO(2,1)$ group does not contain the γ^5 matrix). The subtlety consists in the fact that two different matrices exist that are anticommutating with all the γ^ν-matrices

($\nu = 0, 1, 2$) and with each other for the given four-dimensional representation of the $SO(2,1)$ group. Each of these matrices is suitable for the role of the γ^5-matrix of chiral transformations. Let us remember that the γ^5 matrix given in Section 1 immediately after Eq. (1) is used in this work. Moreover, an alternative form of the γ_5-matrix for the reducible four-dimensional representation of the $SO(2,1)$ group, i.e., $\gamma^5 \equiv \begin{pmatrix} 0, & I \\ I, & 0 \end{pmatrix}$, where $I = 2 \times 2$ is a unit matrix, was used in our recent work [26] that was also devoted to the study of the effect produced by the chemical potentials μ and μ_5 on the superconductivity in (2 + 1)-dimensional GN models, however, in the context of duality between chiral symmetry and superconductivity.

The comparison of our results and the conclusions in [26] shows that superconductivity unavoidably appears in a system with an increase in the chemical potential μ (at any fixed values of the parameters g_1, g_2, and μ_5) independently from the form of the γ^5 matrix. The chiral chemical potential μ_5 has a different property if representation [26] is used for γ^5: in this case, an increase in μ_5 unavoidably leads to the transition of the system into a phase with destroyed chiral symmetry (these properties of chemical potentials μ and μ_5 are particular signs of duality between superconductivity and spontaneous chiral invariance distortion). When the matrix given immediately after the Lagrangian of model (1) is used as γ^5 it can clearly be seen from the results of our work that an increase in μ_5 does not unavoidably lead to the appearance of a phase with destroyed chiral symmetry in a system. Actually, this can be seen from Fig. 2a, which represents the (g_1, g_2)-phase diagram of the model not only at $\mu_5 \neq 0$ and $\mu = 0$, but also at $\mu_5 \to \infty$ and $\mu \neq 0$, that the phase of superconductivity will always persist, e.g., in the region of coupling constants $\{g_{1,2} > 0, g_2 > g_1\}$. Let us also note that we study only one of two types of γ^5-matrices, namely, the type that represents the original result to avoid overloading this paper. The phase diagrams of model (1) with an alternative type of the γ^5-matrix look the same as the diagrams studied in [26].

To summarize this discussion, it is possible to say that the selection of a representation for the γ^5 matrix in (2 + 1)-dimensional GN theories does not have any appreciable effect on the results of this study only in the case of $\mu_5 = 0$, i.e., when the chiral asymmetry of a model is not taken into consideration. In the case of a non-zero μ_5, the selection of a γ^5-matrix may have a substantial effect on the result, as in the studied case of discrete symmetry. This property can be called a characteristic feature of four-fermion models in a (2 + 1) space.

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