ISSN 0027-1349, Moscow University Physics Bulletin, 2017, Vol. 72, No. 4, pp. 340–344. © Allerton Press, Inc., 2017.
Original Russian Text © A.L. Delitsyn, 2017, published in Vestnik Moskovskogo Universiteta, Seriya 3: Fizika, Astronomiya, 2017, No. 4, pp. 12–15.

Determination of Singularities in an Operator of Partial Radiation Conditions

A. L. Delitsyn

Department of Mathematics, Faculty of Physics, Moscow State University, Moscow, Russia
Research and Development Robotics Institute, Ministry of Defense, Moscow, Russia
e-mail: delitsyn@mail.ru

Received May 22, 2016; in final form, December 5, 2016

Abstract—The structure of an operator that determines the partial conditions of radiation in the scalar problem of diffraction theory is considered. Nonlocal boundary conditions are determined by a series setting a certain integro-differential operator. The principal part of this operator is presented in the explicit form of a hyper-singular operator and its components with lower-order singularities. The remaining rapidly converging part of the functional series determines an integral operator with a continuous kernel.

Keywords: diffraction theory, partial conditions of radiation, determination of singularities of an integro-differential operator.

DOI: 10.3103/S002713491704004X

INTRODUCTION

The conditions of unbounded space, in which the diffraction problem is considered, hamper the direct application of the finite element method to the solution. Hence, the integral equation method is widely used to solve this type of problem. At the same time, the integral equation technique used for analysis of the problem of diffraction on a body with variable refraction index requires a large amount of computer time and memory to obtain a more or less realistic solution, since we need to solve huge integral equations.

The problem of diffraction can be reduced to a problem in a finite space with nonlocal boundary conditions, which was accomplished by Sveshnikov in [1–3]. For the scalar problem of diffraction, these conditions were assumed upon the application of mesh methods used to analyze diffraction in waveguides in [4] and diffraction in free space in [5, 6] in combination with the finite-element method. These examples of calculations show an opportunity to develop codes based on the finite-element method for solution of a wide range of problems of diffraction on a multi-body system with a complex shape. The diffusers can be both transparent and nontransparent.

The partial conditions of the radiation are formally similar to the third-type conditions set by the infinite functional series; hence, the problem arises to choose the number of terms that need to be considered during the evaluation of the series. In practice, this problem is often solved empirically on the basis of trial calculations, which is, however, not quite satisfactory. To answer the question about approximation of the operator of the partial conditions of the radiation, we determine the principal part of the functional series, which is set by the hyper-singular integral operator, and then the operators with smaller singularities, until the remaining functional series becomes uniformly converging; here the general term of the series does not exceed the value of the order of $\frac{1}{n^3}$. Replacement of the infinite series with a finite one introduces an error, which is easily evaluated. The singularities of the operator of radiation partial conditions were determined earlier for the boundary problem of Maxwell equations [9]. However, consideration of the scalar problem for the Helmholtz equation has its specifics and leads to operators differing from the ones that were obtained previously. Note that another method is often used in practice: the so-called method of absorbing boundary conditions [7]. However, the error introduced by these conditions is a separate problem due to their approximate character.

STATEMENT OF THE PROBLEM AND THE METHOD OF ITS SOLUTION

Let us consider the boundary problem for the Helmholtz equation

$$\Delta u + k^2 q(x,y,z)u = f(x,y,z), (x,y,z) \in R^3 \qquad (1)$$

with Sommerfeld's radiation conditions

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{r}\right). \tag{2}$$

We assume that $q \equiv 1$, $f \equiv 0$ outside a certain ball with radius R.

Let us recall the statement of radiation partial conditions [1]. It is known [1] that outside the sphere of radius R the solution can be represented in the following form

$$u = \sum_{nm} c_{nm} \frac{H^{(1)}_{n+\frac{1}{2}}(kr)}{\sqrt{r}} Y_n^m(\theta, \phi), \tag{3}$$

where $Y_n^m(\theta,\phi) = P_n^m(\cos\theta)e^{im\phi}$, and c_{nm} are constants. We differentiate the field in the form of series over r and express the unknown constants in terms of parameters of the field on the sphere with radius R as

$$c_{nm} = \frac{(u, Y_n^m)_{L_2(\Omega)}}{H^{(1)}_{n+\frac{1}{2}}(kR) \| Y_n^m \|^2_{L_2}},$$

where Ω is the sphere with radius R, and obtain the known boundary condition [1]:

$$\frac{\partial u}{\partial r} = \sum_{nm} \frac{\left.\frac{d}{dr}\frac{H^{(1)}_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\right|_{r=R}}{\frac{H^{(1)}_{n+\frac{1}{2}}(kR)}{\sqrt{R}}} \frac{(u, Y_n^m)_{L_2(\Omega)}}{\| Y_n^m \|^2_{L_2}}.$$

Hence, the problem in an unlimited space [1] is reduced to a problem in a ball with boundary condition

$$\frac{\partial u}{\partial r} = Hu,$$

where the operator H is determined by series

$$Hu = \sum_{nm} \zeta_n \frac{(u, Y_n^m)_{L_2(\Omega)}}{\| Y_n^m \|^2_{L_2}}, \tag{4}$$

where the coefficients ζ_n are

$$\zeta_n = \frac{\left.\frac{d}{dr}\frac{H^{(1)}_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\right|_{r=R}}{\frac{H^{(1)}_{n+\frac{1}{2}}(kR)}{\sqrt{R}}}.$$

We can prove that coefficients ζ_n increase with n. Consequently, H is not an integral operator. The operator H can be written as a sum of an integro-differential operator and a sum of operators with kernels of decreasing singularity.

We transform ζ_n as follows:

$$\zeta_n = \frac{\left.\frac{d}{dr}\frac{H^{(1)}_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\right|_{r=R}}{\frac{H^{(1)}_{n+\frac{1}{2}}(kR)}{\sqrt{R}}} = -\frac{1}{2R} + k \left.\frac{\frac{d}{dx} H_{n+\frac{1}{2}}(x)}{H_{n+\frac{1}{2}}}\right|_{x=kR}$$

$$\frac{\frac{d}{dx} H_{n+\frac{1}{2}}(x)}{H_{n+\frac{1}{2}}} = -\frac{n+\frac{1}{2}}{x} + \frac{x}{2\left(n-\frac{1}{2}\right)} + R_n.$$

Let us prove that $|R_n| \le \frac{C}{n^3}$. Using recurrent formulas for H_ν [8], we transform $\frac{\frac{d}{dx}H_\nu}{H_\nu}$ to the form:

$$\frac{\frac{d}{dx}H_\nu}{H_\nu} = -\frac{\nu}{x} + \frac{x}{2(\nu-1)\left(1 - \frac{x}{2(\nu-1)}\frac{H_{\nu-2}}{H_{\nu-1}}\right)}$$

$$= -\frac{\nu}{x} + \frac{x}{2(\nu-1)} + R_\nu(x),$$

$$R_\nu = \frac{x}{2(\nu-1)\left(1 - \frac{x}{2(\nu-1)}\frac{H_{\nu-2}}{H_{\nu-1}}\right)} - \frac{x}{2(\nu-1)} =$$

$$-\left(\frac{x}{2(\nu-1)}\right)^2 \frac{H_{\nu-2}}{H_{\nu-1}} \frac{1}{1 - \frac{x}{2(\nu-1)}\frac{H_{\nu-2}}{H_{\nu-1}}} = -\frac{x^2}{8}\frac{1}{(\nu-1)^2(\nu-2)}$$

$$\times \frac{1}{\left(1 - \frac{x}{2(\nu-2)}\frac{H_{\nu-3}}{H_{\nu-2}}\right)\left(1 - \frac{x}{2(\nu-1)}\frac{H_{\nu-2}}{H_{\nu-1}}\right)}.$$

The Nicholson formula [8] implies that $\left|\frac{H_{\nu-1}}{H_\nu}\right| \le 1$, $\forall \nu \ge 1$. As a result, we have:

$$\left(1 - \frac{x}{2(\nu-2)}\frac{H_{\nu-3}}{H_{\nu-2}}\right)\left(1 - \frac{x}{2(\nu-1)}\frac{H_{\nu-2}}{H_{\nu-1}}\right) \ge \frac{1}{C},$$

where C is constant. Consequently $|R_\nu| \le \frac{C}{\nu^3}$. Hence, we can write:

$$\frac{\frac{d}{dx}H_{n+\frac{1}{2}}(x)}{H_{n+\frac{1}{2}}} = -\frac{n+\frac{1}{2}}{x} + \frac{x}{2\left(n-\frac{1}{2}\right)} + R_n,$$

where $|R_n| \le \frac{C}{n^3}$. This implies that the ζ_n coefficients can be presented as

$$\zeta_n = -\frac{n}{R} - \frac{1}{R} + \frac{k^2R}{2\left(n-\frac{1}{2}\right)} + R_n, \quad |R_n| \le \frac{c}{n^3}.$$

Then it is easy to obtain more terms in the ξ_n expansion, which leads to even more complicated formulas for the residual term. What is really important for us is that

$$R_n = \frac{C}{\left(n-\frac{1}{2}\right)^2\left(n-\frac{3}{2}\right)} + \bar{R}_n,$$

where C is constant, and $|\bar{R}_n|$ decreases as $\frac{1}{n^5}$.

Our goal is to obtain the explicit definition of operators that are determined by the principal terms of the series. Let us use the following known formulas

$$\frac{R^2}{4\pi}\frac{\partial}{\partial n_{M_0}}\int\limits_\Omega \frac{\partial}{\partial n_M}\frac{1}{r_{MM_0}}Y_n^m(\theta,\phi)d\Omega = \frac{n(n+1)}{R(2n+1)}Y_n^m(\theta_0,\phi_0). \quad (5)$$

This formula, as presented in [10], provides an opportunity to determine the integro-differential (hyper-singular) operator from (4).

Additionally, we apply the relationship given in [10]

$$\frac{R^2}{4\pi}\int\limits_\Omega \frac{1}{r_{MM_0}}Y_n^m(\theta,\phi)d\Omega = \frac{R}{2n+1}Y_n^m(\theta_0,\phi_0). \quad (6)$$

Equations (5) and (6) allow us to transform the series connected with the principal term $\frac{n}{R}$ in expansion ζ_n in terms of operator H (4) into the explicitly formulated integro-differential and integral operators.

To transform the series connected with term $\frac{k^2R}{2\left(n-\frac{1}{2}\right)}$ we will use the formula

$$-\frac{1}{2\pi R^2}\int\limits_\Omega \ln\sin\frac{\gamma}{2}Y_n^m(\theta,\phi)d\Omega = \frac{1}{n(n+1)}Y_n^m(\theta_0,\phi_0), \quad (7)$$

where $\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)$, γ is the angle between vectors $\mathbf{r}_M$ and $\mathbf{r}_{M_0}$, formula presented in [11] and formula (see [12])

$$\sqrt{\frac{1-x}{2}} = \frac{2}{3} - 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+3)}P_n(x), \quad (8)$$

from which we obtain

$$\sqrt{1-\cos\frac{\gamma}{2}} = \sin\frac{\gamma}{2} = \frac{2}{3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{3}{2}\right)}P_n(\cos\gamma)$$

$$= \frac{2}{3} - \frac{1}{2}\sum_{nm}\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)}\frac{Y_n^m(\phi,\theta)Y_n^m(\theta_0,\phi_0)}{\|Y_n^m\|^2}.$$

We then transform the series defining the operator in the following way. Since

$$\frac{1}{2}n = \frac{n(n+1)}{2n+2} - \frac{n}{2(2n+1)}$$

we get

$$-\sum_{nm}\frac{n}{R}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m = -2\sum_{nm}\frac{1}{R}\frac{n(n+1)}{2n+1}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2} + \sum_{nm}\frac{1}{R}\frac{n}{2n+1}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m.$$

This results in

$$-\sum_{nm}\frac{1}{R}\frac{n(n+1)}{2n+1}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2} = \sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}\frac{R^2}{4\pi}\frac{\partial}{\partial n_0} \times\int\limits_\Omega\frac{\partial}{\partial n}\frac{1}{r_{MM_0}}Y_n^m d\Omega = \frac{R^2}{4\pi}\frac{\partial}{\partial n_0}\int\limits_\Omega\frac{\partial}{\partial n}\frac{1}{r_{MM_0}} \times\sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m d\Omega = \frac{R^2}{4\pi}\frac{\partial}{\partial n_0}\int\limits_\Omega\frac{\partial}{\partial n}\frac{1}{r_{MM_0}}u d\Omega.$$

Hence, the principal part of operator H is determined by integro-differential operator $\frac{R^2}{4\pi}\frac{\partial}{\partial n}\int_\Omega\frac{\partial}{\partial n_0}\frac{1}{r_{MM_0}}u d\Omega$. Further, since

$$\frac{n}{2n+1} = \frac{1}{2} - \frac{n}{2(2n+1)},$$

we get

$$\frac{1}{2R}\sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m = \frac{1}{2R}u.$$

As well,

$$\frac{1}{2R^2}\sum_{nm}\frac{R}{2n+1}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m = \sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}\frac{1}{8\pi}\int_\Omega\frac{Y_n^m}{r_{MM_0}}d\Omega$$
$$= \frac{1}{8\pi}\int_\Omega\frac{Y_n^m}{r_{MM_0}}\sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}d\Omega = \frac{1}{8\pi}\int_\Omega\frac{1}{r_{MM_0}}u\,d\Omega.$$

Since

$$\frac{1}{2n-1} = \frac{2}{2n+1} + \frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)},$$

for term $\frac{1}{2n+1}$ we can derive the already found integral operator.

We still have to consider the series defined by value $\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}$.

In view of the fact that

$$\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)} = \frac{1}{n(n+1)} + \frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)}$$
$$+ \frac{6n+3}{8n(n+1)\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)},$$

we present the operators corresponding to coefficients $\frac{1}{n(n+1)}$ and $\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)}$ as:

$$\frac{1}{4}\sum_{nm}\frac{1}{n(n+1)}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m d\Omega$$
$$= -\frac{1}{8}\frac{1}{\pi R^2}\int_\Omega \ln\sin\frac{\gamma}{2}\sum_{nm}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m d\Omega$$
$$= \frac{1}{8}\frac{1}{\pi R^2}\int_\Omega \ln\sin\frac{\gamma}{2}u\,d\Omega.$$

Further

$$\sum_{nm}\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)}\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m$$
$$= \frac{4}{3}\int_\Omega u\,d\Omega - \frac{1}{R}\int_\Omega r_{MM_0}u\,d\Omega.$$

This kind of additional addend can be obtained upon consideration of the term in expansion ζ_n that decreases with $\frac{1}{n^3}$. Since the calculations are similar to those presented above, we shall omit them in future considerations.

As a result, we obtain the following boundary condition on the surface of the sphere confining the space filled with wave emitters and bodies that are scattering the waves:

$$\frac{\partial u}{\partial r} = C_1\frac{\partial}{\partial n_0}\int_\Omega\frac{\partial}{\partial n}\frac{1}{r_{MM_0}}u\,d\Omega + \frac{1}{2}u$$
$$+ C_2\int_\Omega\frac{1}{r_{MM_0}}u\,d\Omega + C_3\int_\Omega\ln\sin\frac{\gamma}{2}u\,d\Omega \quad (9)$$
$$+ C_4\int_\Omega u\,d\Omega + C_5\int_\Omega r_{MM_0}u\,d\Omega + \sum_{nm}\tilde{R}_n\frac{(u,Y_n^m)_{L_2}}{\|Y_n^m\|^2}Y_n^m,$$

where the coefficients C_i, $i = 1, \ldots, 5$ depend only on k and R and are calculated explicitly. The coefficients $\tilde{R}_n$ decrease no slower than $\frac{1}{n^4}$.

CONCLUSIONS

The results of the application of the finite-element method to the diffraction problem show the opportunity to solve of a wide class of problems using this method with considerably lower requirements for computer time and memory space, compared to the method of integral equations. Use of the radiation partial conditions provides the ability to consider the boundary conditions on a sphere maximally close to a diffuser. At the same time, there is the question of the number of terms of the series, which we need to consider when setting the partial conditions of the radiation. In this paper the operator defining the partial conditions of the radiation is presented in the form of a sum of a hyper-singular operator, a sum of integral operators with kernels of incrementing smoothness, and a residual determined by a rapidly converging series. Application of this operator to solution of the problem of diffraction by mesh methods makes it possible to evaluate the error introduced by the reduction of the infinite series.

REFERENCES

1. A. S. Il'inskii, V. V. Kravtsov, and A. G. Sveshnikov, *Mathematical Models of Electrodynamics* (Vysshaya Shkola, Moscow, 1991).
2. A. G. Sveshnikov, Dokl. Akad. Nauk SSSR **3**, 517 (1950).
3. A. G. Sveshnikov, Dokl. Akad. Nauk SSSR **80**, 345 (1951).

4. A. G. Sveshnikov and A. N. Bogolyubov, Zh. Vychisl. Mat. Mat. Fiz. **14**, 947 (1974).
5. D. A. Konyaev, Moscow Univ. Phys. Bull. **67**, 346 (2012). doi 10.3103/S002713491204008X
6. D. A. Konyaev and A. L. Delitsyn, Mat. Model., No. 8, 48 (2014).
7. M. A. Il'gamov and A. N. Gil'manov, *Non-Reflecting Conditions at the Boundaries of the Region of Computation* (Fizmatlit, Moscow, 2003.
8. G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, 1966).
9. A. L. Delitsyn, Comput. Math. Math. Phys. 49, 1576 (2009).
10. I. K. Lifanov, *Mathematical Models of Electrodynamics: Method of Singular Integral Equations and Numerical Experiment* (Yanus, Moscow, 1995).).
11. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Nauka, Moscow, 1977).
12. N. N. Lebedev, *Special Functions and Their Applications* (Gos. Izd. Fiz.-Mat. Lit., Moscow, 1963).

Translated by N. Semenova