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PHYSICS OF EARTH, ATMOSPHERE, AND HYDROSPHERE

The Effect of the Relativistic Transformation Law of Angles on Laser Ranging of Satellites Moving in Circular Orbits Equipped with a Single Retroreflector

I. V. Mazaeva[a]* and M. A. Pasisnichenko[b]**

[a] *Department of Atmospheric Physics, Faculty of Physics, Moscow State University, Moscow, 119991 Russia*

[b] *Department of Applied Mathematics, Information Technology, and Electrical Engineering, National Research University Moscow Aviation Institute, Moscow, 125993 Russia*

**e-mail: ingamazaeva@yandex.ru*

***e-mail: kaf.pmitet.mai@yandex.ru*

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Abstract—It is shown that due to the relativistic transformation law of angles, a laser pulse reflected from a moving retroreflector propagates not strictly back, but at a small angle to the direction of the laser station. For this reason, the ray located on the periphery of a pulse reaches the receiving telescope of the laser station instead of the central ray of a pulse. As a result, the flux of electromagnetic energy received by the laser station is certainly less than the flux of energy in the vicinity of the central ray. The energy flux attenuation coefficient is assessed on the basis of numerical analysis. It is shown that if the receiving telescope is separated from the laser station in order to be mobile and is moving along the Earth's surface so that the center of each spot formed by a pulse of the reflected light hits the telescope, then the electromagnetic energy flux during laser probing of the satellite will be higher by more than 100 times in comparison with the energy flux received by the stationary telescope of the laser station. From our study it follows that the maximum speed of motion of the centers of spots on the Earth's surface does not exceed 8 km/h.

Keywords: relativistic transformation law of angles, laser probing, ray equation, retroreflector.

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INTRODUCTION

According to the special theory of relativity, the laws of light reflection from a moving mirror differ from the same laws in the case of an immovable mirror. At first sight, this seemingly harmless issue manifests itself very negatively in the laser probing of artificial Earth satellites (AESs). As a rule, retroreflectors installed on the outer panels of an AES have the form of corner reflectors. In the resting frame of reference, the corner reflectors may reflect the light pulses exactly in the direction that they came from. If the aforementioned relativistic effect would be absent, the center of a laser pulse emitted by the telescope of the laser station in the direction of an AES would return back directly to the same telescope after reflection. However, AESs are orbiting in the Earth's gravitation field with appreciable velocities $V_S \sim 5$ km/s. Therefore, according to the general theory of relativity, the coordinates and time in the reference frame with an origin in the center of satellite are related to the coordinates and time of the Earth's observer by more complicated relationships compared to the Lorentz transformations (see [2] for more details). As a result, the direction of a light pulse formed by a retroreflector has a certain angle relative to the initial direction. In [3], calculation showed that in the geocentric nonrotating frame of reference the velocity vector $\mathbf{V}_{\rm ref}$ of the reflected light pulse is given by

$$\mathbf{V}_{\rm ref} = 2\mathbf{V}_{\rm S} - \mathbf{V}_{\rm ph} - 2(\mathbf{V}_{\rm ph}\mathbf{V}_{\rm S})\mathbf{V}_{\rm ph}/c^2 + O(V_{\rm S}^2/c^2)\mathbf{V}_{\rm ph}, \quad (1)$$

where $\mathbf{V}_{\rm S}$ is the AES velocity vector and $\mathbf{V}_{\rm ph}$ is the vector of the momentum of a photon emitted by the laser station.

Unlike a conventional mirror, due to the construction of a moving retroreflector, the maximum angle between the incident and reflected rays is obtained in the case where the reflector moves perpendicular to an incident ray. As a result of this relativistic effect, the center of the reflected light pulse is displaced to a certain distance from the laser station, and the receiving telescope of the station is located at the periphery of this pulse (in the best case). Since the energy flux of electromagnetic radiation in a pulse decreases with the distance from the axis of the pulse to its periphery, the telescope receives only a part of the reflected energy [4]. In particular, when using a single retroreflector, the distribution of the light pulse over its cross section near the Earth's surface is given by the formula [5]

$$\sigma(\eta) = \sigma_0 \left(\frac{2J_1(\eta)}{\eta}\right)^2, \qquad (2)$$

where $J_1(\eta)$ is a Bessel function of zero order, $\eta = \pi D \sin\alpha/\lambda$, α is the angle between the ray under study and the central ray of the reflected beam of light, λ is the wavelength of laser radiation, and D is the output diameter of reflector.

From (2) it follows that the energy flux received by the laser station (at $\eta \neq 0$) is less (Fig. 1) than the energy flux in the center of the spot at $\eta = 0$.

It should also be noted that the centers of spots of the reflected light pulses are not at rest but move over the Earth's surface during laser probing.

The aim of this article is to obtain the energy gain factor of the received light pulses if the receiving telescope is separated from the laser station and moves over the Earth's surface synchronously with the centers of the reflected pulses of light.

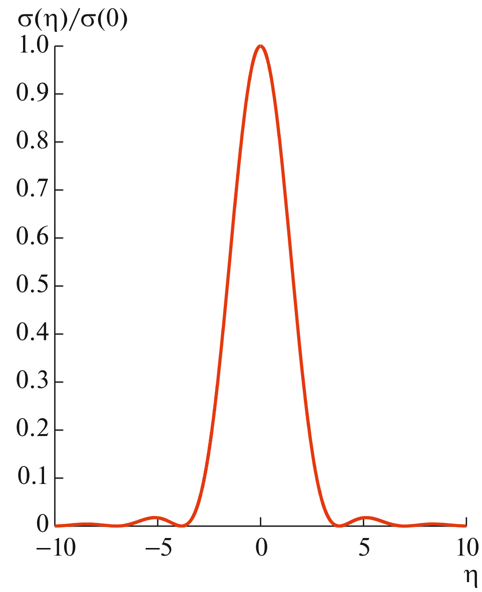

Fig. 1. The magnitude of the received energy flux of the reflected laser pulse depending on the value of parameter η.

1. THE LASER PULSE MOTION FROM A LASER STATION

Consider a laser station located at a point on the Earth's surface with spherical coordinates R_0, θ_0, φ_0. Suppose that the station emits laser pulse in the direction of a satellite equipped with a single retrorflector located in a circular near-earth orbit of radius R_S. The laser pulse is reflected by the retroreflector of this AES at a certain moment of time and returns back to the Earth, forming the spot of radius r_0 on its surface.

Let us find the factor of the amplification of the energy flux received by the mobile telescope of the laser station in comparison to the energy flux received by the stationary telescope. Such calculations in linear [1] and nonlinear [6] electrodynamics are usually carried out by the direct solution of the field equations. However, taking the fact into account that in laser probing the wavelength of electromagnetic radiation ($\lambda = 532$ nm) is much smaller than the satellite's orbit altitude (>300 km), one can use the eikonal approximation.

The calculations will be carried out mainly in the topocentric reference system, the origin of which coincides with the laser station, the *OZ* axis is directed along the local vertical, the *OX* axis is parallel to the meridian, and the *OY* axis is tangential to the parallel. This coordinate system rotates with the Earth relative to the far-distant stars, as a noninertial coordinate system. According to Einstein [7], inertia fields in the noninertial coordinate systems are gravitational fields of a particular type. Therefore, to describe the motion of emitted and reflected laser pulses in the topocentric coordinate system one should use the equations of the general theory of relativity.

Let us write fully covariant Maxwell equations [8] that have the same four-dimensional tensor form in any reference system (and in the presence of a gravitational field)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left\{\sqrt{-g}g^{nm}g^{kl}F_{ml}\right\} = -\frac{4\pi}{c}j^n,$$
$$\frac{\partial F_{nm}}{\partial x^k} + \frac{\partial F_{ml}}{\partial x^n} + \frac{\partial F_{kn}}{\partial x^m} = 0,$$

where g is the determinant of the metric tensor g_{ik} of the pseudo-riemannian space–time and F_{nm} is the electromagnetic field tensor.

From this system of equations it is possible to obtain the eikonal equation using the well-known method [9]

$$g^{nk}\frac{\partial S}{\partial x^n}\frac{\partial S}{\partial x^k} = 0. \qquad (3)$$

Using the method by Lagrange and Sharpi [10], eikonal equation (3) reduces to the equation for an isotropic geodesic line

$$\frac{dk^i}{d\sigma} + \Gamma^i_{nm}k^n k^m = 0, \quad g_{nm}k^n k^m = 0, \qquad (4)$$

where k^n is the tangential four-vector of a geodesic line, σ is an arbitrary affine parameter, and Γ^i_{nm} are the Christoffel symbols.

Equations (4) describe the trajectory of motion of the electromagnetic pulse (ray) in a unified form and the law of its motion along the ray in any pseudo-riemannian space-time [11, 12].

Since the gravitational field of the Earth leads to an insignificant contribution to the ray curvature ($\sim 10^{-3}$ angular sec) [13] that is much less than the contribution of inertia forces [14, 15], we will neglect the contribution of the gravitational field of the Earth in what follows. Then, the metric tensor in the topocentric reference system of the laser station takes the form [16, 17]

$$\begin{gathered} g_{00} = 1 - K^2\{[x\cos\theta_0 + (z + R_0)\sin\theta_0]^2 + y^2\}, \\ g_{01} = Ky\cos\theta_0, \\ g_{02} = -K[x\cos\theta_0 + (z + R_0)\sin\theta_0], \\ g_{03} = Ky\sin\theta_0, \quad g_{11} = g_{22} = g_{33} = -1, \end{gathered} \tag{5}$$

where $K = \Omega/c$, Ω is the Earth's angular rotation frequency.

Substituting expressions (5) into equation (4) and using differentiation over the variable $x^0 = ct$ instead of the differentiation wit respect to the affine parameter σ (as in [18]), we obtain the system of equations

$$\begin{gathered} \ddot{x} - 2K\dot{y}\cos\theta_0 - K^2[x\cos\theta_0 \\ + (z + R_0)\sin\theta_0]\cos\theta_0 = 0, \\ \ddot{y} + 2K[\dot{x}\cos\theta_0 + \dot{z}\sin\theta_0] - K^2 y = 0, \\ \ddot{z} - 2K\dot{y}\sin\theta_0 - K^2[x\cos\theta_0 \\ + (z + R_0)\sin\theta_0]\sin\theta_0 = 0, \end{gathered} \tag{6}$$

where the point denotes the derivative by $x^0 = ct$.

System (6) has the first integral

$$\begin{gathered} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + K^2\{[x\cos\theta_0 \\ + (z + R_0)\sin\theta_0]^2 + y^2\} \\ - 2K\{\dot{x}y\cos\theta_0 - \dot{y}[x\cos\theta_0 \\ + (z + R_0)\sin\theta_0] + \dot{z}y\sin\theta_0\} = 1. \end{gathered} \tag{7}$$

In the topocentric reference system, the coordinates x_S, y_S, z_S of AES have the form [19]

$$\begin{gathered} x_S(t) = R_S\{[\cos(\omega t + \psi_0)\cos(\Omega t + \varphi_0 - \varphi) \\ + \cos\theta\sin(\omega t + \psi_0)\sin(\Omega t + \varphi_0 - \varphi)] \\ \times\cos\theta_0 - \sin\theta\sin\theta_0\sin(\omega t + \psi_0)\}, \\ y_S(t) = R_S\{\cos\theta\sin(\omega t + \psi_0)\cos(\Omega t + \varphi_0 - \varphi) \\ - \cos(\omega t + \psi_0)\sin(\Omega t + \varphi_0 - \varphi)], \\ z_S(t) = R_S\{\sin\theta\cos\theta_0\sin(\omega t + \psi_0) \\ + [\cos(\omega t + \psi_0)\cos(\Omega t + \varphi_0 - \varphi) \\ + \cos\theta\sin(\omega t + \psi_0)\sin(\Omega t + \varphi_0 - \varphi)]\sin\theta_0\} - R_0, \end{gathered} \tag{8}$$

where $\omega = \sqrt{GM/R_S^3}$ is the angular frequency of the AES rotation in the circular orbit, GM is the product of the gravitation constant and the Earth's mass, θ is the orbit inclination angle, φ is the ascending node longitude, and ψ_0 is the angular distance of the AES from the node at the moment $t = 0$.

To decrease the errors of laser probing due to the atmospheric inhomogeneity, it is assumed that the working areas of the celestial sphere are the areas that are 20° above the local horizon. This condition in the topocentric reference system has the form

$$z_S(t) > \sqrt{x_S(t)^2 + y_S(t)^2 + z_S(t)^2}\cos 70^\circ. \tag{9}$$

Therefore, all calculations must be performed only when the AES is located in the region of space that satisfying condition (9).

2. THE MOTION OF A LIGHT PULSE FROM A LASER STATION

Assume that AES occurs in probing area (9) at a certain moment of time t_b. Then, the moment of time at which the laser station emits the Nth light pulse will be determined by the equation $t_N = t_b + N\Delta T$, where ΔT is the time lapse between the two consecutive pulses.

We obtain the equation of a ray along which the light pulse propagates from the laser station to the retroreflector of the AES. For this purpose, we represent the solution of (6) as a bundle of rays that come from the laser station at the moment $t = t_N$ at the point $x = y = z = 0$ of the topocentric reference system:

$$\begin{gathered} x_L(t) = R_0\sin\theta_0\cos\theta_0[\cos\Omega(t - t_N) - 1] \\ + c(t - t_N)\{[n_x\cos[\Omega(t - t_N) + \varphi_0] \\ + n_y\sin[\Omega(t - t_N) + \varphi_0]]\cos\theta_0 - n_z\sin\theta_0\}, \\ y_L(t) = -R_0\sin\theta_0\sin\Omega(t - t_N) \\ - c(t - t_N)[n_x\sin[\Omega(t - t_N) + \varphi_0] \\ - n_y\cos[\Omega(t - t_N) + \varphi_0]], \\ z_L(t) = R_0\sin^2\theta_0[\cos\Omega(t - t_N) - 1] \\ + c(t - t_N)\{[n_x\cos[\Omega(t - t_N) + \varphi_0] \\ + n_y\sin[\Omega(t - t_N) + \varphi_0]]\sin\theta_0 + n_z\cos\theta_0\}, \end{gathered} \tag{10}$$

where n_x, n_y, n_z are the constants of integration for setting the orientation of ray in the space.

Substituting expressions (10) into the first integral (7), we obtain the condition that should be satisfied in this case

$$n_x^2 + n_y^2 + n_z^2 = 1. \tag{11}$$

We choose a ray from bundle of rays (10) that passes through the spatial point $t = t_r$, where the AES is located at this moment. To satisfy this condition, the constants of integration n_x, n_y, n_z and the time t_r should be determined from the equations

$$x_L(t_r) = x_S(t_r), \quad y_L(t_r) = y_S(t_r), \\ \text{and} \quad z_L(t_r) = z_S(t_r).$$

We solve these equations for n_x, n_y, n_z:

$$
\begin{aligned}
n_x &= \frac{1}{c(t_r - t_N)}\{R_S[\cos(\Omega t_N - \varphi)\cos(\omega t_r + \psi_0) \\
&\quad + \sin(\Omega t_N - \varphi)\sin(\omega t_r + \psi_0)\cos\theta] \\
&\quad - R_0 \sin\theta_0 \cos\varphi_0\}, \\
n_y &= \frac{1}{c(t_r - t_N)}\{R_S[\cos(\Omega t_N - \varphi) \\
&\quad \times\sin(\omega t_r + \psi_0)\cos\theta \\
&\quad - \sin(\Omega t_N - \varphi)\cos(\omega t_r + \psi_0)] - R_0 \sin\theta_0 \sin\varphi_0\}, \\
n_z &= \frac{R_S \sin\theta \sin(\omega t_r + \psi_0) - R_0 \cos\theta_0}{c(t_r - t_N)}.
\end{aligned}
\tag{12}
$$

Substituting (12) into (11) we arrive to the equation for determining the moment t_r:

$$
\begin{aligned}
c^2(t_r - t_N)^2 &= R_S^2 - 2R_0 R_S \\
&\times\{\sin\theta_0[\cos(\omega t_r + \psi_0)\cos(\Omega t_N + \varphi_0 - \varphi) \\
&+ \cos\theta \sin(\omega t_r + \psi_0)\sin(\Omega t_N + \varphi_0 - \varphi)] \\
&+ \cos\theta_0 \sin\theta \sin(\omega t_r + \psi_0) + R_0^2.
\end{aligned}
$$

When exchanging by the laser pulses from an AES situated in the orbit with a radius less than the Earth–Moon distance, the assessment $\omega(t_r - t_N) \sim 10^{-5}$ holds. Due to this, this equation can be expanded in a series by this small parameter for all AESs. With an accuracy of $\sim 10^{-7}$ we obtain the equation

$$
\begin{aligned}
c^2(t_r - t_N)^2 - L^2 &= 2c(t_r - t_N)\frac{\omega}{c} R_S R_0 \\
&\times\{\sin\theta_0[\sin(\omega t_N + \psi_0)\cos(\Omega t_N + \varphi_0 - \varphi) \\
&- \cos\theta \cos(\omega t_N + \psi_0)\sin(\Omega t_N + \varphi_0 - \varphi)] \\
&- \cos\theta_0 \sin\theta \cos(\omega t_N + \psi_0)\} = 0,
\end{aligned}
$$

where we used the definition

$$
\begin{aligned}
L^2 &= R_S^2 + R_0^2 - 2R_S R_0\{\sin\theta_0 \\
&\times[\cos(\omega t_N + \psi_0)\cos(\Omega t_N + \varphi_0 - \varphi) \\
&+ \cos\theta \sin(\omega t_N + \psi_0)\sin(\Omega t_N + \varphi_0 - \varphi)] \\
&+ \cos\theta_0 \sin\theta \sin(\omega t_N + \psi_0)\}.
\end{aligned}
$$

Solving this equation for t_r, one can find the moment of reflection of a light pulse that was emitted by the laser station at the moment t_N

$$
\begin{aligned}
t_r &= t_r(t_N) = t_N + \frac{L}{c} + \frac{\omega}{c^2} R_0 R_S \\
&\times\{[\sin(\omega t_N + \psi_0)\cos(\Omega t_N + \varphi_0 - \varphi) \\
&- \cos\theta \cos(\omega t_N + \psi_0)\sin(\Omega t_N + \varphi_0 - \varphi)] \\
&\times \sin\theta_0 - \cos\theta_0 \sin\theta \cos(\omega t_N + \psi_0)\}.
\end{aligned}
\tag{13}
$$

Thus, expressions (10), (12), and (13) describe the motion of a light pulse along the ray from the laser station to an AES in a circular orbit in the topocentric reference system.

3. LASER PULSE REFLECTION FROM A MOVING REFLECTOR

Let us calculate all vectors in expression (1). Formulas for transforming the coordinates x, y, z in the topocentric reference system into the x^{Geo}, y^{Geo}, z^{Geo} coordinates in a nonrotating geocentric frame of reference have a rather complicated form [2, 19]. Neglecting in these formulas the contributions that are insignificant in our problem one obtains

$$
\begin{aligned}
x^{\mathrm{Geo}} &= [x\cos\theta_0 + (z + R_0)\sin\theta_0] \\
&\times \cos(\Omega t + \varphi_0) - y\sin(\Omega t + \varphi_0), \\
y^{\mathrm{Geo}} &= [x\cos\theta_0 + (z + R_0)\sin\theta_0] \\
&\times \sin(\Omega t + \varphi_0) + y\cos(\Omega t + \varphi_0), \\
z^{\mathrm{Geo}} &= -x\sin\theta_0 + (z + R_0)\cos\theta_0.
\end{aligned}
$$

Using (8) we find the law of motion of AESs in a circular orbit in a nonrotating geocentric reference system:

$$
\begin{aligned}
x_{\mathrm{S}}^{\mathrm{Geo}}(t) &= R_{\mathrm{S}}[\cos\varphi \cos(\omega t + \psi_0) \\
&- \sin\varphi \cos\theta \sin(\omega t + \psi_0)], \\
y_{\mathrm{S}}^{\mathrm{Geo}}(t) &= R_{\mathrm{S}}[\sin\varphi \cos(\omega t + \psi_0) \\
&+ \cos\varphi \cos\theta \sin(\omega t + \psi_0)], \\
z_{\mathrm{S}}^{\mathrm{Geo}}(t) &= R_{\mathrm{S}} \sin\theta \sin(\omega t + \psi_0).
\end{aligned}
$$

After differentiating these expressions with respect to time and substituting $t = t_r$ we obtain the components of the AES velocity vector $\mathbf{V}_r$ in this reference system

$$
\begin{aligned}
\mathbf{V}_r^x(t_r) &= -\omega R_{\mathrm{S}}[\cos\varphi \sin(\omega t_r + \psi_0) \\
&+ \sin\varphi \cos\theta \cos(\omega t_r + \psi_0)], \\
\mathbf{V}_r^y(t_r) &= \omega R_{\mathrm{S}}[\cos\varphi \cos\theta \cos(\omega t_r + \psi_0) \\
&- \sin\varphi \sin(\omega t_r + \psi_0)], \\
\mathbf{V}_r^z(t_r) &= \omega R_{\mathrm{S}} \sin\theta \cos(\omega t_r + \psi_0).
\end{aligned}
\tag{14}
$$

In a completely analogous way, one obtains ray equation (10) in a non-rotating frame of reference (for $t_N \le t \le t_r$)

$$
\begin{aligned}
x_L^{\mathrm{Geo}}(t) &= R_0 \sin\theta_0 \cos(\Omega t_N + \varphi_0) + c(t - t_N)M_x, \\
y_L^{\mathrm{Geo}}(t) &= R_0 \sin\theta_0 \sin(\Omega t_N + \varphi_0) + c(t - t_N)M_y, \\
z_L^{\mathrm{Geo}}(t) &= R_0 \cos\theta_0 + c(t - t_N)M_z,
\end{aligned}
$$

where the components of vector **M** in these formulas have the form

$$\mathbf{M}_x = \frac{1}{c(t_r - t_N)}\{R_S[\cos\varphi\cos(\omega t_r + \psi_0) - \sin\varphi\cos\theta\sin(\omega t_r + \psi_0)] - R_0\sin\theta_0\cos(\Omega t_N + \varphi_0)\},$$
$$\mathbf{M}_y = \frac{1}{c(t_r - t_N)}\{R_S[\sin\varphi\cos(\omega t_r + \psi_0) + \cos\varphi\cos\theta\sin(\omega t_r + \psi_0)] - R_0\sin\theta_0\sin(\Omega t_N + \varphi_0)\},$$
$$\mathbf{M}_z = \frac{1}{c(t_r - t_N)}\{R_S\sin\theta\sin(\omega t_r + \psi_0) - R_0\cos\theta_0.$$

Now we calculate the velocity vector $\mathbf{V}_{\rm ph}$ of the center of laser pulse in this reference system at the moment $t = t_r$

$$\mathbf{V}_{\rm ph}^x(t_r) = \left.\frac{dx_L^{\rm Geo}(t)}{dt}\right|_{t=t_r} = cM_x,$$
$$\mathbf{V}_{\rm ph}^y(t_r) = \left.\frac{dy_L^{\rm Geo}(t)}{dt}\right|_{t=t_r} = cM_y, \quad (15)$$
$$\mathbf{V}_{\rm ph}^z(t_r) = \left.\frac{dz_L^{\rm Geo}(t)}{dt}\right|_{t=t_r} = cM_z.$$

The components of velocity vector $\mathbf{V}_{\rm ph}$ of a reflected laser pulse in the nonrotating geocentric coordinate system can be obtained if expressions (14) and (15) are substituted into (1).

4. CALCULATING THE ANGLE BETWEEN THE CENTRAL RAY AND THE RAY CONNECTING THE RETROREFLECTOR AND THE LASER STATION

Upon being reflected from the retroreflector, a light pulse propagates in the form of a divergent beam of rays, one of which connects the AES and the laser station. This ray forms a definite angle α with the central ray of the reflected pulse. To find the sinus of this angle, we first find the equation of the ray that connects the retroreflector with the laser station at $t \geq t_r$, written in a nonrotating geocentric reference system. Solving equations (6) with the initial condition $x_L^{\rm Geo}(t_r) = x_S^{\rm Geo}(t_r)$, $y_L^{\rm Geo}(t_r) = y_S^{\rm Geo}(t_r)$, $z_L^{\rm Geo}(t_r) = z_S^{\rm Geo}(t_r)$, we write the equation of a bundle of rays of the reflected light pulse that emanate from this point:

$$x_{\rm ref}^{\rm Geo}(t) = x_S^{\rm Geo}(t_r) + \mathbf{L}_x c(t - t_r),$$
$$y_{\rm ref}^{\rm Geo}(t) = y_S^{\rm Geo}(t_r) + \mathbf{L}_y c(t - t_r), \quad (16)$$
$$z_{\rm ref}^{\rm Geo}(t) = z_S^{\rm Geo}(t_r) + \mathbf{L}_z c(t - t_r),$$

where **L** is a constant vector, which due to (7) must be a unit vector

$$\mathbf{L}_x^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2 = 1. \quad (17)$$

The coordinates of the laser station in a nonrotating geocentric frame of reference are the following functions of time

$$x_{\rm las}^{\rm Geo}(t) = R_0\sin\theta_0\cos(\Omega t + \psi_0),$$
$$y_{\rm las}^{\rm Geo}(t) = R_0\sin\theta_0\sin(\Omega t + \psi_0),$$
$$z_{\rm las}^{\rm Geo}(t) = R_0\cos\theta_0.$$

From a bundle of rays we choose the ray that connects the retroreflector and the laser station. It is necessary to require that at certain moment t^* of time this ray passes through the laser station:

$$x_{\rm ref}^{\rm Geo}(t^*) = x_{\rm las}^{\rm Geo}(t^*), \quad y_{\rm ref}^{\rm Geo}(t^*) = y_{\rm las}^{\rm Geo}(t^*),$$
$$z_{\rm ref}^{\rm Geo}(t^*) = z_{\rm las}^{\rm Geo}(t^*).$$

From which we find the components of vector **L**

$$\mathbf{L}_x = \frac{1}{c(t^* - t_r)}\{R_0\cos(\Omega t^* + \varphi_0)\sin\theta_0 - R_S[\cos(\omega t_r + \psi_0)\cos\varphi - \cos\theta\sin(\omega t_r + \psi_0)\sin\varphi]\},$$
$$\mathbf{L}_y = \frac{1}{c(t^* - t_r)}\{R_0\sin(\Omega t^* + \varphi_0)\sin\theta_0 - R_S[\cos(\omega t_r + \psi_0)\sin\varphi + \cos\theta\sin(\omega t_r + \psi_0)\cos\varphi]\},$$
$$\mathbf{L}_z = \frac{R_0\cos\theta_0 - R_0\sin\theta\sin(\omega t_r + \psi_0)}{c(t^* - t_r)}.$$

Then, from (17) one obtains a transcendental equation for determining the moment t^*

$$c^2(t^* - t_r)^2 = R_S^2 + R_0^2 - 2R_0R_S \times\{\sin\theta_0[\cos(\omega t_r + \psi_0)\cos(\Omega t^* + \varphi_0 - \varphi) + \cos\theta\sin(\omega t_r + \psi_0)\sin(\Omega t^* + \varphi_0 - \varphi)] + \cos\theta_0\sin\theta\sin(\omega t_r + \psi_0)\}.$$

Expanding this equation in a series by the small parameter $\Omega(t^* - t_r) \sim 10^{-5}$ s, with quadratic accuracy we obtain

$$c^2(t^* - t_r)^2 - 2c(t^* - t_r)\frac{\Omega}{c}R_SR_0 \times\sin\theta_0\{\cos(\omega t_r + \psi_0)\sin(\Omega t_r + \varphi_0 - \varphi) - \cos\theta\sin(\omega t_r + \psi_0)\cos(\Omega t_r + \varphi_0 - \varphi)\} - R_{0r}^2 = 0,$$

Here, we used for brevity the definition

$$R_{0r} = R_S^2 + R_0^2 - 2R_SR_0 \times\{\sin\theta_0[\cos(\omega t_r + \psi_0)\cos(\Omega t_r + \varphi_0 - \varphi) + \cos\theta\sin(\omega t_r + \psi_0)\sin(\Omega t_r + \varphi_0 - \varphi)] + \cos\theta_0\sin\theta\sin(\omega t_r + \psi_0)\}$$

for the squared distance between the laser station and the point of the AES localization at the moment $t = t_r$. With sufficient accuracy, the solution of the above equation takes the form

$$c^2(t^* - t_r) = R_{0r} + \frac{\Omega}{c}R_0R_S \times\sin\theta_0[\cos(\omega t_r + \psi_0)\sin(\Omega t_r + \varphi_0 - \varphi) - \cos\theta\sin(\omega t_r + \psi_0)\cos(\Omega t_r + \varphi_0 - \varphi)].$$

In a nonrotating geocentric reference system, the light propagates rectilinearly and the central ray of the reflected laser pulse is directed along the velocity vector $\mathbf{V}_{\text{ref}}$, starting at the point $\mathbf{r} = \mathbf{r}_S^{\text{Geo}}(t_r)$ at $t = t_r$. Then, the law of motion of the center of the reflected light pulse in this reference system takes the form

$$x_{\text{ref}}^{\text{Geo}}(t) = x_S^{\text{Geo}}(t_r) + V_{\text{ref}}^x(t_r)(t - t_r),$$
$$y_{\text{ref}}^{\text{Geo}}(t) = y_S^{\text{Geo}}(t_r) + V_{\text{ref}}^y(t_r)(t - t_r), \quad (18)$$
$$z_{\text{ref}}^{\text{Geo}}(t) = z_S^{\text{Geo}}(t_r) + V_{\text{ref}}^z(t_r)(t - t_r).$$

Now, we find the angle α between the central ray of the reflected light pulse and the ray connecting the retroreflector and the laser station. This angle coincides with an angle between the tangents to these rays [20] at the point $\{x_S^{\text{Geo}}(t_r), y_S^{\text{Geo}}(t_r), z_S^{\text{Geo}}(t_r)\}$ in which the retroreflector is located at the moment of light reflection. As follows from (16) and (18), the vector tangent to the ray connecting the retroreflector and the laser station coincides with vector $\mathbf{L}$, whereas the tangent vector to the central ray coincides with vector $\mathbf{V}_{\text{ref}}$.

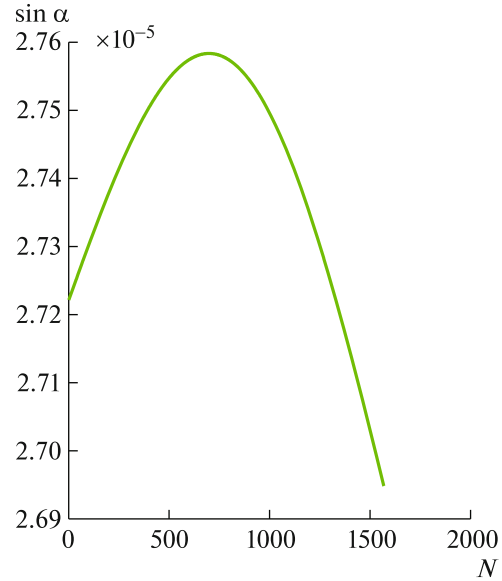

Fig. 2. The change in the value of sinα depending on the pulse number *N* of probing, when the AES is moving in the region satisfying condition (9).

With allowance for the fact that the vector $\mathbf{L}$ is a unit vector, and the modulus of $\mathbf{V}_{\text{ref}}$ equals the speed of light (with necessary accuracy), the $\sin\alpha$ can be represented in the form

$$\sin\alpha = \frac{\|\mathbf{L}\mathbf{V}_{\text{ref}}\|}{|\mathbf{L}||\mathbf{V}_{\text{ref}}|} = \sqrt{\frac{(L_xV_y^{\text{ref}} - L_yV_x^{\text{ref}})^2 + (L_xV_z^{\text{ref}} - L_zV_x^{\text{ref}})^2 + (L_zV_y^{\text{ref}} - L_yV_z^{\text{ref}})^2}{c^2}}. \quad (19)$$

Since the analytical expressions for the components of vectors in (16) are very complicated, further calculations are carried out numerically.

5. THE RESULTS OF NUMERICAL CALCULATIONS

The formulas we obtained make it possible to investigate the change in the angle between the direction from which the light pulse arrives to the AES reflector and the direction of its reflection.

We assume that the satellite is equipped with a prismatic corner reflector (manufactured by JSC SMC SPP) with a diameter of the inscribed circle D = 27 mm [21] and that the wavelength of laser radiation used is $\lambda = 532$ nm. The distribution of the energy flux of the reflected light pulse along its cross section is described by formula (2). As example AESs, we chose two satellites: the high-orbiting Ethalon-2 (the orbit radius is 25 498 km and the inclination angle is 65.5°) and the low-orbiting JASON-2 (the orbit radius is 7714 km and the inclination angle is 66°), both of which have corner laser reflectors on their outer panels. We also assume that laser probing of these AESs is performed from the Svetloye station [22] (60.5332° northern latitude and 29.7805° eastern longitude; the height above the geoid is 69 m).

Numerical calculation showed that the value of $\sin\alpha$ determined by relation (19) when Ethalon-2 AES is located in region (9), first increases from 2.72×10^{-5} to 2.76×10^{-5} and then decreases to 2.7×10^{-5} (Fig. 2). This corresponds to a change in the parameter η from 4.34 to 4.4 and then to 4.3.

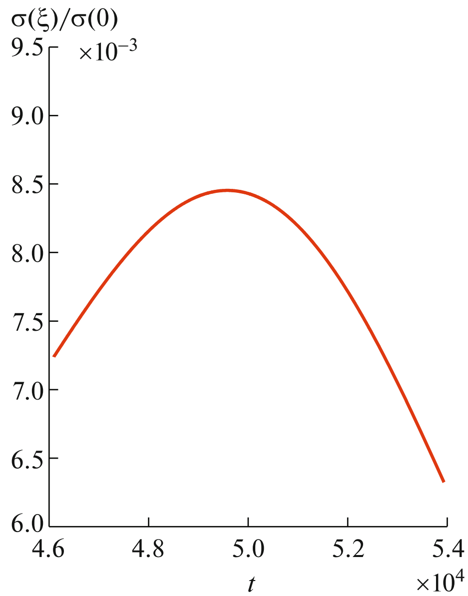

Fig. 3. The temporal dependence of the reduction factor ξ of the received energy flux of a pulse reflected at the point of laser station localization in comparison with the flux at the center of the spot (time t in conds).

Thus, when probing this satellite, the laser station falls into the region of the graph shown in Fig. 1, between the first zero and the second maximum.

When probing the low-orbiting Jason-2 AES, $\sin\alpha$ increases from 4.6×10^{-5} to 4.75×10^{-5} and then decreases to 4.5×10^{-5}. This corresponds to changing the parameter η from 7.3 to 7.64 and then to 7.2. Thus, when probing this satellite, the laser station falls in the region of the graph shown in Fig. 1 between the second maximum and the second zero. The larger value of $\sin\alpha$ for low-orbiting AESs, compared with the high-orbiting AESs, is associated with a greater orbital speed of the first satellite compared to the second one. Actually, from formula (1) it follows that in the nonrotating geocentric reference frame the $\sin\alpha$ reaches the maximum value $2V_S/c$. Since the orbital velocity of an AES is inversely proportional to the square root of the orbit radius, one obtains $\sin\alpha \sim 1/\sqrt{R_0}$. The transition from a nonrotating geocentric reference frame to the topocentric reference frame rotating with the Earth also contributes to the value of $\sin\alpha$.

We now obtain an assessment of the ratio between the energy flux of the reflected light pulse received by an immovable telescope of laser station to the energy flux received by a mobile telescope that follows the centers of the reflected light spots on the Earth's surface. Let the reduction factor be $\xi = \sigma(\eta)/\sigma(0)$. This factor shows by how many times the energy flux of the reflected light pulse received by the telescope of an immovable laser station is lower than the energy flux received by a mobile telescope that follows the centers of the spots.

Numerical calculation showed that for probing Ethalon-2, the factor ξ changes within the limits from 6.5×10^{-3} to 8.5×10^{-3} (Fig. 3).

Since the change in the parameter η when probing high-orbiting and low-orbiting satellites coincides in the order of magnitude, then the change in the coefficient ξ for them also coincides in the order of magnitude.

Therefore, the use of a mobile receiving telescope that follows behind the centers of the spots of the reflected light pulses on the Earth's surface will make it possible to increase the value of the received energy flux in comparison with the stationary telescope of the laser station by more than 100 times.

As follows from numerical assessments, the maximum velocity of such a telescope on the Earth's surface does not exceed 8 km/h.

CONCLUSIONS

The investigation showed that due to the relativistic transformation of angles, an orbiting reflector does not reflect laser pulses strictly in the back direction, but at a certain angle to it. Therefore, the laser station is located not in the center of the reflected spot, but on its periphery.

The energy flux received by the AES reflectors can be increased by more than 100 times if a receiving telescope is separated from the laser station, is movable, and is moving along the Earth's surface so that the center of each spot formed by the reflected light hits the telescope. The maximum speed of motion of such a telescope along the Earth's surface does not exceed 8 km/h, which is quite realizable technically. With such a method of probing the satellite, it will be necessary to change the receiving and transmitting antenna, the registration system, and the software. However, such modifications are of a technical nature and can be performed at the current level of technical development. As a result, the area of confident registration of the reflected pulses during laser probing of AESs will be significantly wider.

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