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PHYSICS OF EARTH, ATMOSPHERE, AND HYDROSPHERE

On the Asymptotics of Multidimensional Linear Wave Packets: Reference Solutions

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Abstract—The classic problem of linear wave-packet propagation in a dispersive medium is considered. Asymptotic equations of the Cauchy problem for two-dimensional Gaussian wave packets are constructed in terms of Fourier integrals. These asymptotic solutions are regular at the caustics and describe new physical features of wave-packet propagation: rotation in space and formation of a wave front with anomalously slow dispersion (quasi-dispersive).

Keywords: linear waves, wave dispersion, method of stationary phase, method of steepest descent, saddle-point method, wave packet dispersion.

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INTRODUCTION

The problem of linear dispersive wave propagation is recognized as well studied. An impressive arsenal of mathematical tools is based on representation of the general solution in the form of expansion over eigenfunctions of the problem, which allows the solution of various problems of wave propagation in inhomogeneous and nonstationary media. Asymptotic stationary phase and saddle-point methods have been proposed as long ago as in the 19th century (P.S. Laplace, G.G. Stokes, and W. Kelvin) as approaches essentially considering wave properties: the oscillating character of solutions and/or the solution localization in physical space or in the space of wave vectors. These methods provide an opportunity to analyze the principal physical effect that determines the evolution of linear wave packets: wave dispersion.

The case of one-dimensional wave propagation is of special interest, since it makes it possible to formulate some important results in the form of explicit analytical expressions. These are primarily the results concerning asymptotical behavior of wave packets for long periods of time or at large distances from an emitter. The asymptotic analysis forecasts the amplitude decrease as the square root of time in the coordinate system bound to the maximum of a wave disturbance for waves of an arbitrary physical nature, except for special cases of degeneration of the dispersion ratio, namely, that of the second derivative of the frequency over the wave number. The natural solution of the degeneration problem by taking higher differentials of dispersion dependence into consideration provides a solution in terms of the Airy function and slower dispersive decay of the wave amplitude as $A \sim t^{-1/3}$.

Generalization of asymptotic methods to a multidimensional case is quite often made "by analogy," without due consideration of the multidimensionality of the problem. Transformation of the coordinate system is often considered to provide an opportunity to reduce the problem to a one-dimensional problem with consequent following of the formal pattern of the one-dimensional problem solution. Degeneration of a quadratic form related to the second differential is treated in the same way: the eigenvector related to the zero eigenvalue determines the direction in which the problem can be reduced to a one-dimensional problem with customary solutions in the form of the Airy function. In many wave problems, this simplistic approach can be viewed as acceptable and mathematical rigor seen as an excess for the set tasks. Below, we show that mathematically strict treatment applied even in the case of the two-dimensional problem reveals important physical effects and that the involvement of higher (cubic) terms of asymptotic expansion is not necessary even at degeneration of the quadratic form. The solution appears not to be the power dependence $A \sim t^{-1/3}$, but $A = \text{const}$.

Geophysical hydrodynamics offers the richest spectrum of problems of wave propagation in a medium with anisotropy related to effects of rotation and stratification. In these problems, correct application of asymptotic methods reveals many remarkable effects determined by the multidimensionality of the wave packet and volume finiteness. In this paper, the main effects related to the multidimensionality and examples in which these effects occur are considered. The paper begins a series of studies of multidimensionality effects in wave problems.

1. ASYMPTOTIC METHODS IN THE THEORY OF PROPAGATION OF LINEAR POINT WAVE PACKETS

We consider the classical problem of development of a disturbance in an anisotropic spatially homogeneous medium allowing the existence of small amplitude waves. Here, the cases of medium spatial inhomogeneity and nonstationarity are not considered and will be presented in subsequent papers. Let wave emitters and attenuation be absent, and the initial disturbance be localized in a certain physical space (which tends to zero rapidly at infinity). Let us suppose that we have managed to reduce the problem to a multidimensional Fourier integral:

$$f_n(\mathbf{x},t) = (2\pi)^{-n/2}\int F_n(\mathbf{k})\exp(i\mathbf{kx} - i\omega(\mathbf{k})t)d\mathbf{k}, \quad (1)$$

where n is the spatial dimension, $F_n(\mathbf{k})$ is the Fourier image of function f_n at moment $t = 0$, $\mathbf{k}$ is the multidimensional wave vector, and $\omega(\mathbf{k})$ is the dispersion relationship determined by the type of considered small-amplitude waves and assumed to be known. The real part of (1) $\mathrm{Re} f_n(\mathbf{x}, t)$ is the solution of the physical problem.

Apparently, integral (1) cannot be calculated analytically for the general case. At the same time, integrals of the (1) type allow the efficient use of asymptotic methods. Presently, mathematical tools of the asymptotic methods have been well stated in many papers [1–4]. The multidimensional method of the stationary phase (see Lighthill's classical book [4]) reflects these method specifics most brightly. The major parameter (time or distance) determines the calculation of integral (1) over a rapidly oscillating function. The highest term of the corresponding asymptotic expansion is related to contributions of the bending point neighborhoods and phase stationarity (thus the name of the method). The Taylor series expansion of phase $\theta = \mathbf{kx} - \omega(\mathbf{k})t$ accurate within quadratic terms appears sufficient to calculate the highest term of the expansion. The corresponding quadratic form is used one way or another in many asymptotic approaches (for example, the classical stationary phase method [2] and Maslov's canonical operator method [1]).

The presence of formally large parameter in phase θ (1) means slow modification of the function $F_n(\mathbf{k})$, i.e., strong localization of the wave function in physical space. In this case, it is makes sense to speak about point-wave packets and pointwise asymptotics of solutions of (1). In this case, expansion of the phase function $\theta = \mathbf{kx} - \omega(\mathbf{k})t$ is connected with the maximum point of a wave disturbance in physical space. With degeneration of the quadratic form, a singularity in an asymptotic expression emerges at the thus-selected point.

In this paper, we are guided by different physical premises and propose the method of reference solutions. We assume the packet to be localized in the wave vector space, and it means that we consider the wave packets of finite sizes weakly modulated in physical space. Such an approach also leads to the analysis of behavior of the quadratic form at expansion of phase function $\theta = \mathbf{kx} - \omega(\mathbf{k})t$, but not in relation to the current maximum of the wave amplitude, but regarding the carrier harmonic of the modulated wave packet. Speaking about the reference functions and solutions, we only change the method of description of the wave packet kinematics passing from the wave function in the coordinate space (analogy to Euler hydrodynamics description) to the wave packet–quasi-particle marked with the wave vector of the carrier wave harmonic (the Lagrangian approach). Note that a similar approach applying reference (model) Gaussian packets is widely used, for example, in problems of radio physics and fiber optics. The issue of the validity range, which is very important for both approaches, is not considered here. The above-stated physical arguments appear sufficient for a wide range of problems of wave propagation.

We consider a Taylor expansion for the frequency $\omega(\mathbf{k})$ in the neighborhood of point $\mathbf{k}_0 = (k_0, l_0)$:

$$\begin{aligned}\omega(k,l) &= \omega(k_0,l_0) + \omega'_k(k-k_0) + \omega'_l(l-l_0)\\ &+ 1/2\omega''_{kk}(k-k_0)^2 + 1/2\omega''_{ll}(l-l_0)^2\\ &+ \omega''_{kl}(k-k_0)(l-l_0) + \dots\end{aligned} \quad (2)$$

Let us put the real symmetric two-dimensional matrix and characteristic equation [5] in correspondence to the quadratic part of phase θ expansion in (1)

$$\mathbf{A}_2 = \begin{Vmatrix} tk_0^2\omega''_{kk} & tk_0l_0\omega''_{kl} \\ tk_0l_0\omega''_{kl} & tl_0^2\omega''_{ll}\end{Vmatrix};$$
$$\det(\mathbf{A}_2 - \lambda_{1,2}\mathbf{E}) = \det\begin{Vmatrix} tk_0^2\omega''_{kk} - \lambda_{1,2} & tk_0l_0\omega''_{kl} \\ tk_0l_0\omega''_{kl} & tl_0^2\omega''_{ll} - \lambda_{1,2}\end{Vmatrix} = 0. \quad (3)$$

Expanding the determinant, we gain the following quadratic equation for eigenvalues

$$\lambda_{1,2}^2 - \lambda_{1,2}\mathrm{Tr}(\mathbf{A}_2) + \det(\mathbf{A}_2) = 0. \quad (4)$$

The coefficients in (4) are the determinant $\det(\mathbf{A}_2)$ and matrix track $\mathrm{Tr}(\mathbf{A}_2)$ and they are orthogonal invariants of the matrix $\mathbf{A}_2$

$$\det(\mathbf{A}_2) = t^2 k_0^2 l_0^2[\omega''_{kk}\omega''_{ll} - (\omega''_{kl})^2];$$
$$\mathrm{Tr}(\mathbf{A}_2) = t[k_0^2\omega''_{kk} + l_0^2\omega''_{ll}]. \quad (5)$$

Note that the Euler curvature is the track of the matrix $\mathrm{Tr}(\mathbf{A}_2)$ [6]. By virtue of the symmetry and reality of the matrix $\mathbf{A}_2$, eigenvalues $\lambda_{1,2}$ are also real. We highlight a simple, but important fact: calculation of the roots in (4) is not mandatory for evaluation of integral (1) by the stationary phase method. The wave-packet amplitude is determined by product $\lambda_1\lambda_2$, i.e., according to Vieta's theorem, by determinant $\mathbf{A}_2$. The asymptote of integral (1) $f_2(\mathbf{x}, t) \approx (i\lambda_1\lambda_2)^{-1/2} \approx (i\det\mathbf{A}_2)^{-1/2}$ has a simple physical sense: the product $\lambda_1\lambda_2$ is proportional to the physical volume of the wave packet (λ_1 and λ_2 set the principal axes of quadratic form $\mathbf{A}_2$). The imaginary unit sets the phase shift $\pm\pi/4$; the sign of this shift is determined by the condition of solution attenuation at infinity and can be obtained by passing to the complex plane and applying various modes of closure of the integration contour. In a particular case, it leads to a Fourier integral, which was first calculated by Euler [7]. The introduction of low attenuation and bypass of the pole over the complex plane leads to the saddle-point method (see, for example, [4]) and naturally provides the same result.

Generalization to the case of arbitrary dimensionality of space n does not produce special difficulties. The asymptote of integral (1) has the form

$$f_n(\mathbf{x},t) \approx F(\mathbf{k}_0)[\det\mathbf{A}_n]^{-1/2}$$
$$\times \exp[i(\mathbf{k}_0\mathbf{x} - \omega(\mathbf{k}_0)t)]\exp\left[\frac{i\pi}{4}\mathrm{sgn}\,\mathbf{A}_n\right] \quad (6)$$

in the limit $t \to +\infty$; $|x_i| \to +\infty$. The wave vector $\mathbf{k}_0$ is set by the stationary condition

$$\frac{x_i}{t} = \frac{\partial\omega}{\partial k_i}, \quad (7)$$

while condition

$$\det\left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right) \neq 0 \quad (8)$$

determines the sufficiency of expansion (2) accurate within the quadratic, by small deviations of the wave vector terms. In Eq. (6), $\mathrm{sgn}\mathbf{A}_n$ is the matrix signature, the difference between the number of positive and negative eigenvalues of matrix $\mathbf{A}_n$.

Asymptotic equation (6) is interpreted as follows. Assuming large t, x_i and finite ratio x_i/t, we solve ray equations (7) in relation to wave numbers $k_i = k_i(x_i/t)$. The obtained wave vector $\mathbf{k}_0(\mathbf{x}/t)$ in (6) describes the rescaling (change of length or period) of the wave-packet carrier in the process of its propagation. Note that obtaining $\mathbf{k}_0(\mathbf{x}/t)$, i.e., a reversion of the ray equations, can be nontrivial, even for the waves in a homogeneous and isotropic medium, not to mention more general situations.

Asymptotic equations (6)–(8) can be termed dispersive: at long times and a nonzero determinant (8), each spatial coordinate shows attenuation $t^{-1/2}$, and the wave amplitude decreases in inverse proportion to the wave packet "volume" in n-dimension space, i.e., as $t^{-n/2}$. Speaking about the "volume" of the wave packets, we should keep in mind the conditionality of this term, namely: we consider the asymptotic behavior of a point-wave packet, whose distribution is described by a delta function in the space of wave numbers. This is the factor that creates the amplitude of the Fourier-harmonic $F_n(\mathbf{k}_0)$ that occurs in (6) at stationarity point (7). Below, we use expansion (2) for the alternative method of evaluation of the integral (1) asymptotic equation. The alternative is actually reduced to a different definition of the vector $\mathbf{k}_0 = (k_0, l_0)$.

The transfer to the problem of wave packets of a finite volume seems clear: it is necessary to introduce space–time modulation of the wave packet into consideration and take the effects that occur into account. One of the ways to do this is to formally introduce a small attenuation for these point-wave packets. The final solution is obtained in the limit of infinitesimal attenuation. The problem with passage to the limit appears unsolvable at caustics related to a singularity of the asymptotic expression, when the determinant of matrix $\mathrm{sgn}\mathbf{A}_n$ becomes zero. At such points, the known asymptotic methods, including Maslov's canonical operator method, face difficulties, which are solved by construction of new asymptotic solutions. These problems are solved by analogy to a one-dimensional problem taking the cubic term in the frequency expansion into account. The obtained solutions in the form of Airy functions are connected by asymptotic solutions far from the caustic. This problem appears to be quite time consuming.

Below we show how the finiteness of the physical volume of the wave packet can be taken into account by "smearing" the solution across the space of wave vectors. In drastic contrast to other methods, our approach does not require fulfillment of stationarity condition (7); this allows us to avoid solving the ray equations, which is indispensable for point-wave packets. As well, we avoid problems at caustics with determinant $\mathbf{A}_n$ (8) turning to zero. This is achieved at the cost of introducing constraints on the initial conditions: we are limited to the reference forms of wave packets. In our opinion, this physically obvious approach cannot significantly distort the transformation of wave packets of small, but finite volume, primarily, considering the wave-packet rotation effect.

2. PHYSICAL EFFECTS OF PROPAGATION OF WAVE PACKETS OF FINITE VOLUME

Let a wave packet have a Gaussian distribution at initial moment $t = 0$

$$f_2(x,y,t=0) = \frac{1}{\Delta x \Delta y} \times \exp\left[ik_0 x + il_0 y - \frac{x^2}{2(\Delta x)^2} - \frac{y^2}{2(\Delta y)^2}\right] \quad (9)$$

and the Fourier image

$$F_2(k,l) = \exp\left[-\frac{(k-k_0)^2}{2(\Delta k)^2} - \frac{(l-l_0)^2}{2(\Delta l)^2}\right], \quad (10)$$

where $\Delta k = (\Delta x)^{-1}$, $\Delta l = (\Delta y)^{-1}$ describe the wave packet width in the coordinate and the Fourier spaces. Hereinafter, the corresponding solutions for distributions (9) and (10) will be named reference solutions. In the limit $\Delta k \to 0$, $\Delta l \to 0$, distribution (10) tends to a δ function; however, integral (1) essentially differs from the above-presented pointwise asymptotics (6) and (7). It is very important that we do not have a necessity for passage to the limit!!!

For the integrand in (1), we construct symmetric complex two-dimensional matrix $\mathbf{B}_2$ in the quadratic approximation

$$\mathbf{B}_2 = \left\| \begin{matrix} 1 + it(\Delta k)^2\omega''_{kk} & it\Delta k\Delta l\omega''_{kl} \\ it\Delta k\Delta l\omega''_{kl} & 1 + it(\Delta l)^2\omega''_{ll} \end{matrix} \right\|, \quad (11)$$

which differs from matrix $\mathbf{A}_2$ in (3) by having unity in its diagonal. Another important distinction: in $\mathbf{A}_2$, we used the carrier wave number only for the reasons of dimensionality, this scale did not render any influence on the subsequent calculations. Introduction of new spatial scales of the width of the wave packet Δk, Δl removes this degeneration that is present in pointwise solutions.

The behavior of reference solutions is still determined by invariants of quadratic form (11); however, these invariants have a form that is more complex:

$$\det \mathbf{B}_2 = 1 + it[(\Delta k)^2\omega''_{kk} + (\Delta l)^2\omega''_{ll}] + t^2(\Delta k)^2(\Delta l)^2[(\omega''_{kl})^2 - \omega''_{kk}\omega''_{ll}], \quad (12)$$

$$\mathrm{Tr}(\mathbf{B}_2) = it[(\Delta k)^2\omega''_{kk} + (\Delta l)^2\omega''_{ll}] + 2.$$

These dependences are not reduced to products of combined second derivatives of the dispersion dependence (compare to (4)), which implies the absence of wave packet scale. Accordingly, amplitude A does not become zero at degeneration of the quadratic form that is determined only by the wave dispersion. The more general appearance of an argument of the exponent containing linear terms does not essentially complicate the problem. The Fourier transformation of the quadratic form exponent (see, for example, [2]) gives us the following expression

$$\int_{R^n} \exp\left[-\frac{1}{2}\langle \mathbf{B}x, x\rangle - i\langle x, \xi\rangle\right] dx = (2\pi)^{n/2}(\det \mathbf{B})^{-1/2} \exp\left[-\frac{1}{2}\langle \mathbf{B}^{-1}\xi, \xi\rangle\right]. \quad (13)$$

Here, $\mathbf{B}^{-1}$ is an inverse matrix. For solutions that decrease at infinity we have

$$\sqrt{\det \mathbf{B}^{-1}} = |\det \mathbf{B}|^{-1/2} \exp\left[-\frac{1}{2}\arg \mathbf{B}\right].$$

For initial conditions (10), the real part of matrix $\mathbf{B}$ is strictly positive ($\mathrm{Re}\mathbf{B} > 0$), which ensures the absolute convergence of the integral. In our case, a dimensionless argument of quadratic form $\xi(\xi_x, \xi_y)$ appears in (13):

$$\xi_x = \Delta k(x - c_{gr.x}t); \quad \xi_y = \Delta l(y - c_{gr.y}t), \quad (14)$$

determining the deviation from the wave-packet trajectory with the wave vector of the carrier harmonic $\mathbf{k}_0$. Here, $c_{gr.x} \equiv \omega'_k$; $c_{gr.y} \equiv \omega'_l$ in (14) are components of the group velocity.

It is necessary to note the important physical distinctions of the considered approach from the stationary phase method: transfer to parameters like (14) $\xi(\xi_x, \xi_y) = 0$ is provided by the presence of large parameters: "infinite" time or distance (see (6)). In our approach to reference solutions, the smallness of argument $\xi(\xi_x, \xi_y)$ determining the parameter deviation is related to the narrowness of the wave-packet spectrum ($\Delta k \to 0$, $\Delta l \to 0$). There are no explicit constraints on either coordinates or time in this approach. In this sense, it is possible to consider our approach as more general than the customary asymptotic methods.

For the determinant of matrix $\mathbf{B}$ in the neighborhood of the carrier wave vector $\mathbf{k}_0 = (k_0, l_0)$, we can write:

$$D^2 = |\det \mathbf{B}|^2 = [1 + t^2(\mu_{xy}^2 - \mu_x\mu_y)]^2 + t^2(\mu_x + \mu_y)^2, \quad (15)$$

where the following designations are used for derivatives of the dispersion relationship:

$$\mu_x = (\Delta k)^2\omega''_{kk}, \quad \mu_y = (\Delta k)^2\omega''_{ll}, \quad \mu_{xy} = \Delta k\Delta l\omega''_{kl}.$$

The wave function then has the form

$$f_2(x,y,t) = \Psi_2(x,y,t)\cos[\phi(x,y,t)] \quad (16)$$

with the envelope (at $t = 0$ it is Gaussian distribution)

$$\Psi_2(x,y,t)=\frac{1}{\sqrt{D}}\exp\left[\frac{\xi_x^2(1+t^2(\mu_{xy}^2+\mu_y^2))+\xi_y^2(1+t^2(\mu_{xy}^2+\mu_x^2))-2\xi_x\xi_y t^2\mu_{xy}(\mu_x+\mu_y)}{2D^2}\right] \quad (17)$$

and phase function

$$\phi(x,y,t)=k_0x+l_0y-\omega(k_0,l_0)t-\theta_2(t)-\omega_2(x,y,t). \quad (18)$$

The correction to the carrier frequency $\omega(k_0, l_0)$ is set by the quadratic form

$$\begin{aligned}\omega_2(x,y,t)=2D^2[&\xi_x^2(t^2\mu_y(\mu_{xy}-\mu_x\mu_y)-t\mu_x)\\&+\xi_y^2(t^2\mu_x(\mu_{xy}-\mu_x\mu_y)-t\mu_y)\\&+2\xi_x\xi_y t\mu_{xy}(1+t^2(\mu_{xy}-\mu_x\mu_y))],\end{aligned} \quad (19)$$

and the phase shift also depends on time

$$\theta_2=\frac{1}{2}\arctan\left[\frac{(\mu_x+\mu_y)t}{1+t^2(\mu_{xy}-\mu_x\mu_y)}\right]. \quad (20)$$

Solution (14)–(20) is true at $t \geq 0$, and at moment $t = 0$ it coincides with (8) and describes dispersion smearing accompanied by rotation of the wave packet of finite width by a certain angle in physical space. Apparently, the rate of the wave packet rotation is finite and asymptotically tends to zero at $t \to +\infty$.

The numerator of the exponent index (17) can be rewritten in the following form:

$$\begin{aligned}C(\xi_x,\xi_y)=-t^2[&(\xi_x\mu_{xy}-\xi_y\mu_x)^2\\&+(\xi_y\mu_{xy}-\xi_x\mu_y)^2]-\xi_x^2-\xi_y^2,\end{aligned} \quad (21)$$

illustrating the nature of temporal transformation of the initial (time-independent part) shape of the wave packet. The time-dependent part is an ellipse with axes $\xi_x\mu_{xy}-\xi_y\mu_x=0$, $\xi_y\mu_{xy}-\xi_x\mu_y=0$. At longer times, it becomes determining. Interesting enough, the initially isotropic distribution does not mean the absence of the wave-packet rotation effect: at $t \to +\infty$, the shape of the wave packet is determined both by its initial width (Δk, Δl), and the properties of wave dispersion (μ_x, μ_y, μ_{xy}). The constructed solution (16)–(20) does not formally require solving the ray equations, unlike the above-considered pointwise asymptotics. The wave vector of the carrier harmonic $\mathbf{k} = (k_0, l_0)$ is fixed, which drastically simplifies the solution.

In the important special case of $\omega''_{kl} = 0$ and an isotropic dispersion relationship, the group velocity direction coincides with one of the ellipse axes. No packet rotation occurs. Matrix $\mathbf{B}$ and inverse matrix $\mathbf{B}^{-1}$ are diagonal, and two-dimensional solution (17) is split into the product of two one-dimensional dispersive solutions:

$$\begin{aligned}&\Psi_2(x,y,t)=\Psi(x,t)\Psi(y,t),\\&\Psi(x,t)=\frac{1}{(1+t^2\mu_x^2)^{1/4}}\exp\left[\frac{(x-c_{gr.x}t)^2}{2(\Delta x)_0^2/(1+\mu_x^2t^2)}\right],\\&\Psi(y,t)=\frac{1}{(1+t^2\mu_y^2)^{1/4}}\exp\left[\frac{(y-c_{gr.y}t)^2}{2(\Delta y)_0^2/(1+\mu_y^2t^2)}\right].\end{aligned} \quad (22)$$

The phase correction is split into a sum of arctangents [8]

$$\begin{aligned}\theta_2&=-\frac{1}{2}\arctan\left[t\frac{\mu_x+\mu_y}{1-t^2\mu_x\mu_y)}\right]\\&=\frac{1}{2}[\arctan(t(\Delta k)^2\omega''_{kk})+\arctan(t(\Delta l)^2\omega''_{ll})].\end{aligned} \quad (23)$$

At $t \to \infty$, (23) gives the limiting values of phase $\theta_2(\infty) = \pm\pi/2$ or 0, depending on the signs of the derivatives ω_{kk} and ω_{ll}. In the n-dimensional case, the difference between the numbers of positive and negative second derivatives corresponds to the real matrix signature. Hence, for θ_2 in (23), we have a correct limit process to the point-packet asymptotic. Note that (22) and (23) types of multidimensional solutions are well known [9].

Qualitative distinctions between pointwise asymptotic and asymptotic equations for wave packets of finite sizes are discovered at fulfillment of condition

$$\Delta_{kl}=(\omega''_{kl})^2-\omega''_{kk}\omega''_{ll}=0 \quad \text{at} \quad \omega''_{kl}\neq 0. \quad (24)$$

For pointwise asymptotic equations, this situation cannot be resolved by phase expansion to within quadratic terms in the neighborhood of the stationary phase point. The higher-order expansions lead to solutions in terms of the Airy function. However, there is no problem in considering wave packets of finite sizes: in the index of the exponential envelope $\Psi_2(x, y, t)$, the terms highest according to time fold up into a perfect square and (23) describes the degeneration of constant phase elliptical lines into a straight line in the limit $t \to \infty$. In one direction, the size of the wave packet tends to the limiting value, while in direction normal to this one, it extends to infinity. In this case, the solution can be named quasi-dispersive, since dispersive smearing in the two-dimensional space is sim-

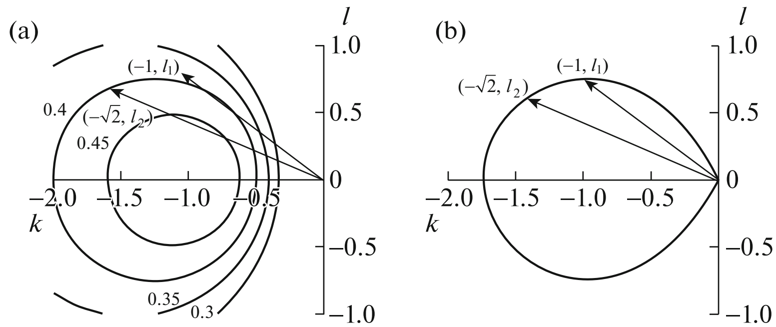

Fig. 1. (a) Isofrequencies of dispersion relationship (26) in dimensionless coordinates $\tilde{\mathbf{k}} = (\tilde{k}, \tilde{l})$. Isofrequencies $\tilde{\omega} = 0.45, 0.4, 0.35, 0.3$, and wave vectors $\tilde{\mathbf{k}}_1 = (-1, l_1)$, $l_1 = \sqrt{\sqrt{17/4} - 3/2}$; $\tilde{\mathbf{k}}_2 = (-\sqrt{2}, l_2)$ are shown for examples displayed in Figs. 2–4. (b) The condition of dispersion degeneration (24) for baroclinic Rossby wave dispersion equation (26). Vectors for the examples displayed in Figs. 2–4 are shown.

ilar to the one-dimensional case. From (17)–(20), we have

$$\xi_0 = x_c\sqrt{|\omega''_{ll}|} - y_c\sqrt{|\omega''_{kk}|}\,\mathrm{sgn}[\omega''_{kl}(\omega''_{kk} + \omega''_{ll})],$$
$$x_c = x - c_{gr.x}t, \quad y_c = y - c_{gr.y}t,$$
$$\Psi_2(x, y, t) = \frac{1}{D^{1/2}} \tag{25}$$
$$\times\exp\left[-\frac{t^2(\Delta k)^2(\Delta l)^2|\mu_x - \mu_y|\xi_0^2 - (\xi_x^2 + \xi_y^2)}{2D^2}\right].$$

The isolines of solution (25) are ellipses, and their eccentricity and orientation of principal axes depend on time. Condition (24) stipulates that these ellipses degenerate into a set of parallel straight lines at $t \to +\infty$. Any cross section of the wave packet is of Gaussian form, but the width varies with time differently: along the minor axis it tends to a constant value, while along the major axis, it smears linearly with time, which provides wave-packet attenuation similar to the one-dimensional case $D^{-1/2} \sim t^{-1/2}$. Apparently, our approach, which takes the finite sizes of the wave packet into consideration in the simplest possible way causes qualitatively new effects (rotation of the wave-packet and modification of its shape) in comparison with the one-dimensional case. At the same time, our approach removes the difficulty related to degeneration of determinant (24) that describes the dispersion of pointwise solutions irrespective of the higher terms of asymptotic expansions.

Note that for the isotropic dispersion law $\omega(k, l) = \Omega(\varrho)$, $\varrho = (k^2 + l^2)$, the condition of caustic $(\omega''_{kl})^2 - \omega''_{kk}\omega''_{ll} = 0$ is reduced to $\Omega'(\Omega' + 2\varrho\Omega'') = 0$ and is realized only for linear dependence $\Omega = (a\varrho^{1/2} + b)$ (a and b are random constants).

3. EXAMPLE. A ROSSBY WAVE PACKET OF FINITE SIZE

In conclusion, we present an example to illustrate the considered asymptotic analysis of the evolution of two-dimensional linear wave packets of finite sizes. We consider linear Rossby waves in the ocean with permanent stratification, the dispersion relationship for which looks like

$$\omega = \frac{-\beta k}{k^2 + l^2 + a^2}. \tag{26}$$

Here, $\beta = 2\Omega\cos\theta_0/R$ is the northern gradient of the Coriolis parameter at latitude θ_0, R is the radius, Ω is the Earth's rotation frequency, $a^2 = \pi^2 n^2\Omega^2/(N^2H^2)$ is the square of the eigenvalue of the boundary problem along the vertical coordinate, $N^2 = -\frac{g}{\rho}\frac{\partial p}{\partial z}$ is the Brunt–Väisälä frequency, which is assumed to be constant, H is the depth of the ocean, k is the zonal component of the wave vector, and l is the meridional component of the wave vector, and $n = 0, 1, \ldots$ is the ordinal number of the mode. For the barotropic mode $n = 0$, scale a becomes zero. Condition (24) on the degeneration of the point-wave packet dispersion in terms of dimensionless wave numbers $\tilde{k} = k/a$, $\tilde{l} = l/a$ leads to the equation of hyperbolic lemniscates (a special case of Booth's lemniscate), which re well known in the theory of plane algebraic curves

$$\tilde{k}^4 + 2\tilde{k}^2\tilde{l}^2 + \tilde{l}^4 - 3\tilde{k}^2 + \tilde{l}^2 = 0. \tag{27}$$

This curve, which is universal for the considered type of waves, is displayed in Fig. 1b compared with dispersion relationship (26) for baroclinic Rossby waves (Fig. 1a). It is shown that the effects that are of interest to us occur, for example, for waves with rather high frequencies (vectors $\mathbf{k}_1$ and $\mathbf{k}_2$) in the neighborhood of the maximum dimensionless frequency $\tilde{\omega}_{\max} = 1/2$.

The lemniscate splits the entire phase space of Rossby waves into two zones. Inside the curve (the waves with the highest frequency), determinant $\mathbf{B}_2$ is positive, i.e., the dispersion is positive. The external zone corresponds to negative dispersion. The sign of the dispersion essentially determines the dynamics of weakly nonlinear dispersive waves [11]. In particular, generation of cyclones and anticyclones, which are often interpreted as nonlinear Rossby waves, is naturally related to the corresponding zones of wave numbers represented by the lemniscate (cyclones correspond to positive dispersion that takes place inside the lemniscate, anticyclones correspond to negative dispersion that occurs outside the lemniscate). The correct consideration of this issue requires the analysis of three-dimensional Rossby waves, which will be presented in our future papers.

Let us consider the effect of quasi-dispersion of a packet of two-dimensional Rossby waves with the dimensionless carrier wave vector $\tilde{\mathbf{k}}_1$ = $(-1, [(17/4)^{1/2} - 3/2]^{1/2})$. The anisotropy of the dispersion relationship makes possible this nontrivial effect manifestation even for the initially isotropic shape of the wave packet. In Figs. 2–4, isolines of the wave packet envelope are shown at different moments. At shorter times ($t = 1$, $t = 10$), evolution is reduced to slow, almost isotropic, smearing of the wave packet. At longer times ($t = 100$, $t = 1000$), the smearing is prolonged according to a law that is close to linear (attention to the scale of axes). The smearing becomes strongly anisotropic: the wave packet is stretched out in a certain direction, asymptotically tending to the limit at infinite times. Apparently, a qualitatively similar effect of strong anisotropization of the initially symmetric shape of the wave packet will be observed for similar wave vectors as well. In other words, this effect is structurally stable and rough for the finite zone of the space of wave numbers and the considered type of waves (Rossby waves), and consequently, is physically significant. Within the limits of the proposed approach, the problem of structural stability is solved rather simply. Note that at construction of pointwise asymptotics of wave packets close to caustics, i.e., while considering higher approximations of the method of stationary phase, the issue of the roughness of the respective solutions remains open in many cases.

In conclusion, we present two more cases of Rossby wave packet behavior at the critical value of the dimensionless wave vector $\tilde{\mathbf{k}}_1$ = $(-2^{1/2}, [(33^{1/2} - 5)/2])$. In Figs. 3 and 4, as in Fig. 2, the isolines of the wave-packet envelope are shown at moments $t = 0$, 5, 25, and 125 for initial distributions in the form of an ellipse. In Fig. 3, an ellipse with the principal axes ratio of 2 is stretched along the x-axis, and changes its shape and orientation during its evolution. The evolution lasts for significantly shorter periods than in the case of Fig. 2, and the wave packet rotation takes less time than the subsequent quasi-dispersion smearing. The evolution of the wave packet with the same parameters, but stretched along the ordinate axis (Fig. 4) is similar.

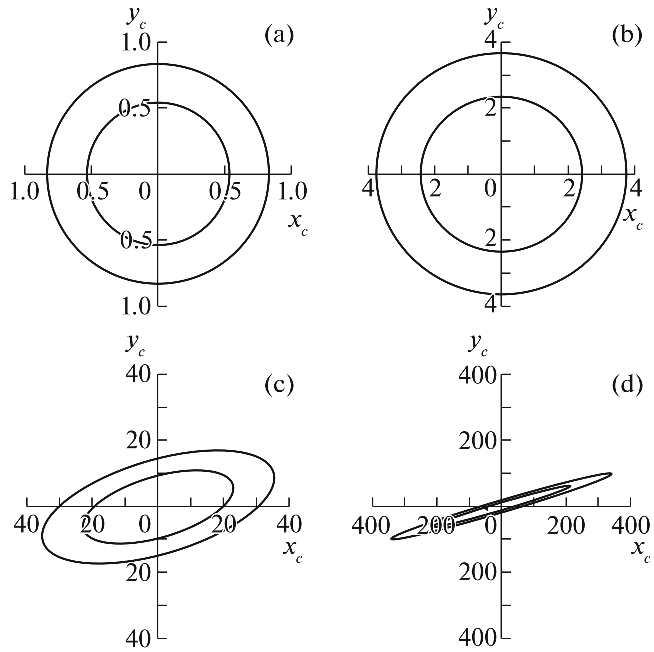

Fig. 2. The shape of a Rossby wave packet with dimensionless wave vector $\tilde{\mathbf{k}}_1 = (-1, \sqrt{\sqrt{17/4} - 3/2})$ at different moments. Isolines correspond to levels 0.75 and 0.5 of the wave maximum amplitude at $t = 1$, 10, 100, and 1000.

We complete the series of examples with Fig. 5, where the wave-packet cross sections at the level of the half-maximum are shown, corresponding to the case of Fig. 3 for times $t = 0$, 5, 25, and 125.

CONCLUSIONS

We have considered the approximate solutions of some problems of wave propagation in a multi-dimensional medium in view of the finiteness of the physical volume of a wave packet. Our approach should not be considered as a criticism of the well-developed asymptotic methods in the theory of linear wave propagation. The known asymptotic methods (stationary phase, saddle point, etc.) implicitly contain an assumption about the infinitesimal volume of a wave packet. The problem of the singularity of these asymptotic equations in the case of degeneration of dispersion is well understood by specialists and is definitely related to the pointwise representation of wave packets [3]. We introduced wave-packet modulation, i.e., we consider wave packets of finite volumes from the very beginning. This statement raises serious difficulties for classical asymptotic methods. A possible dispute of the

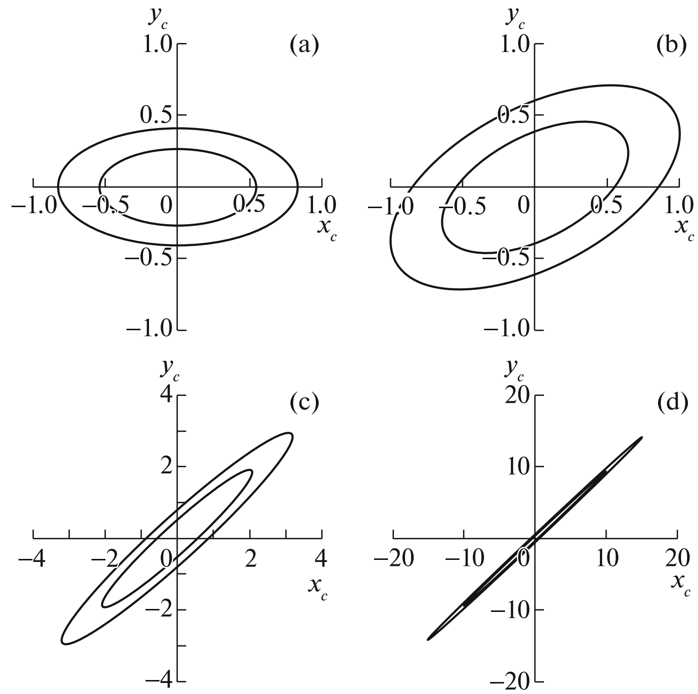

Fig. 3. The shape of a Rossby wave packet for the dimensionless wave vector $\tilde{\mathbf{k}}_2 = (-\sqrt{2}, \sqrt{(-5+\sqrt{33})2})$ at $t = 0$, 5, 25, and 125. The isolines correspond to the 0.75 and 0.5 levels of the maximum amplitude of the wave.

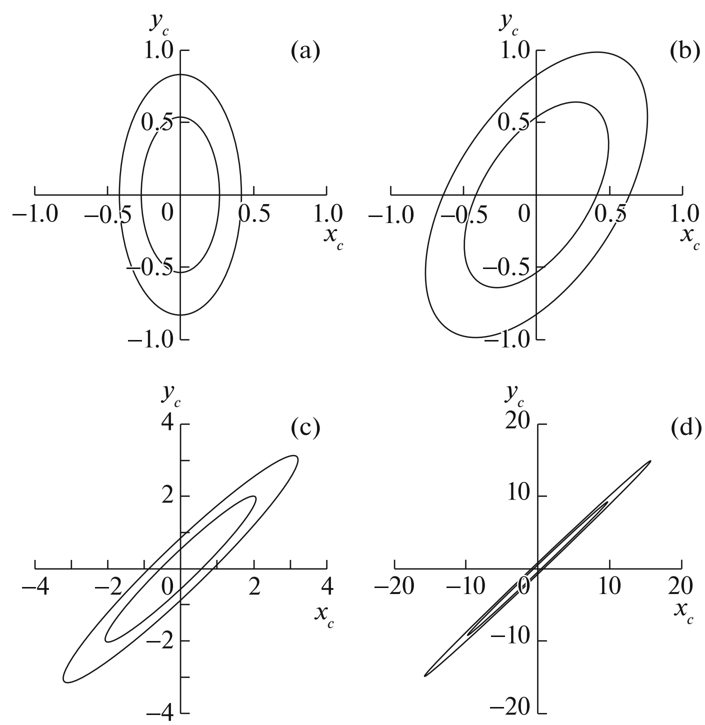

Fig. 4. The shape of a Rossby wave packet for the dimensionless wave vector $\tilde{\mathbf{k}}_2 = (-\sqrt{2}, \sqrt{(-5+\sqrt{33})2})$ at different moments $t =$ 0, 5, 25, and 125. The initial contour of the wave packet is oriented at right angles to the one shown in Fig. 3. The isolines correspond to the 0.75 and 0.5 levels of the maximum amplitude of the wave.

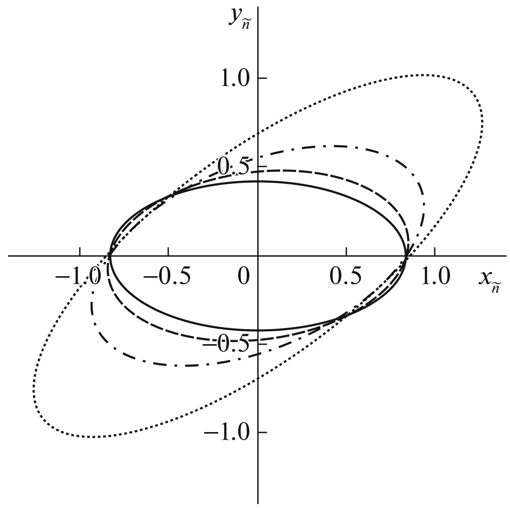

Fig. 5. Wave packet cross sections at a level of the half-maximum corresponding to the case shown in Fig. 3 $\tilde{\mathbf{k}}_2 = (-\sqrt{2}, \sqrt{(-5+\sqrt{33})2})$: for times $t = 0$ is solid line; $t = 5$ is the dashed line; $t = 25$ is the dashed-and-dotted line; $t = 125$ is the dotted line.

validities of various approaches faces the problem of physical realities: how valid it is to consider a wave packet as a point in coordinate space or in momentum space. For oceanic waves, in our opinion, the resolution of these disputes can occur through reconciliation of both points of view: the range of probable conditions is so wide that we need to consider all of the physical effects predicted by these approaches.

Our rather rough routine approach provides the greatest advantages in the multi-dimensional case, when the evolution of a wave packet gains an additional degree of freedom: an opportunity to modify the packet shape and orientation of its envelope. We consider it excessive to discuss the many physical effects related to this quite simple fact. The strong anisotropy of wave motions in the ocean inspires many theories. We show that one explanation is related to the intrinsic kinematics of packets of linear dispersive waves. In our future publications we plan to present examples of various wave types.

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