A STUDY OF LINEAR SYSTEMS WITH MANY DELAYS BY THE ROOT-LOCUS METHOD

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The general properties of the root-locus plot of a transcendental characteristic equation of the form

$$\Phi_1(p)\,e^{p\tau_1}+\ldots+\Phi_k(p)\,e^{p\tau_k}+\Lambda(p)+$$
$$+\rho\left[\Psi_1(p)\,e^{p\tau_1}+\ldots+\Psi_k(p)\,e^{p\tau_k}+\Theta(p)\right]=0,$$

where ρ is a free parameter, are examined. Using these properties, it is possible to influence the dynamics of complex systems with delay significantly by varying the positions of the initial and limit points and the asymptotic forms of the root loci. An analytic root-locus equation that permits efficient use of digital computers for analysis and synthesis of such systems is derived.

Despite the progress made in recent years in the theory of differential equations with deviating arguments, engineering methods of computation are still limited to comparatively simple systems with delay. The D-partition method is essentially the only one that is relatively universal both in the sense of the structural complexity of the system and with respect to the number of delay elements. At the same time, the root-locus method does not embody any fundamental limitations in these respects either. In our view, however, it has advantages in system synthesis, since it can be used not only to investigate stability and obtain the critical parameter values at which the system is excited and the frequencies at which this occurs, but also to establish how the properties of the system will vary on changes in its parameters and introduction of correcting blocks or couplings into the system. Its lucidity and high information content are attracting increasing attention from investigators. It is no accident that this was the method used in several recent papers dealing with more or less complex systems with delay [1,2].

However, the theory of the root-locus method has been elaborated in detail [3] only for systems that can be reduced to single-loop systems with a single delay in the free parameter (the gain). The paper [4] defends the use of the method for both single-loop and multiloop systems with a single delay block for an arbitrarily chosen linear free parameter. Below we indicate the possibilities of the root-locus method for study of systems containing arbitrary numbers of delay elements that cannot be reduced to an equivalent block the delay in which is equal to the sum of all the delays.

In general, the natural motions in such systems are described by a stationary linear differential equation with a deviating argument of the form

$$\sum_{i=0}^{k}\sum_{j=0}^{n}a_{ij}\,x^{(j)}(t-\tau_i)=0.$$

The solution of this equation is sought [5], as for ordinary differential equations, in the form $x = e^{pt}$. The resulting characteristic equation is multiplied by $e^{p\tau_{max}}$ and write it in the form

$$\Phi_1^0(p)\,e^{p\tau_1}+\ldots+\Phi_k^0(p)\,e^{p\tau_k}+\Lambda^0(p)=0, \qquad (1)$$

where $\Phi_1^0, \ldots, \Phi_k^0, \Lambda^0$ are polynomials of $p = \delta + j\omega$, and $\tau_1, \ldots, \tau_k$ are the delay times. For the sake of argument, we shall henceforth assume that the condition $\tau_1>\tau_2>\ldots>\tau_k>\tau_0=0$ is

satisfied. Having chosen the free parameter ρ, as a function of which it is necessary to investigate the motion of the roots of the characteristic equation on the complex p-plane, we rewrite this equation in the form

$$\Phi_1(p)\,e^{p\tau_1} + \ldots + \Phi_k(p)\,e^{p\tau_k} + \Lambda(p) + \rho\,[\Psi_1(p)\,e^{p\tau_1} + \ldots + \Psi_k(p)\,e^{p\tau_k} + \Theta(p)] = 0. \qquad (2)$$

We observe that ρ can be any parameter that enters into the equation linearly, e.g., the gain. It was shown by L. S. Pontryagin [6] that the solution of the differential equation can be stable only if the corresponding transcendental characteristic equation has a principal term. It will be recalled that the principal term of (1) will be $\Phi_i^0(p)\,e^{p\tau_i}$ if the degree of the polynomial Φ_i^0 is larger than the degrees of the other polynomials and τ_i is larger than all other τ. For practical purposes, therefore, there is no point in discussing equations that do not contain a principal term, although this is not a limitation for the construction of root loci.

Let us establish the basic properties of the root loci of (2). As we know [7], some of these properties yield much information on the behavior of the system even without recourse to construction of the root hodograph. Most important among these properties are the positions of the initial and limit points and the asymptotic form of the root loci.

The root portrait of the system is fully determined by the relative positions of the initial (ρ = 0) and limit ($\rho \to \pm\infty$) points. The equations for their determination are obtained from (2) with the corresponding values of the free parameter. It is clear that these equations themselves have the form of (1), but are usually less complex (lower order of polynomial, fewer terms). Each of these equations can be investigated in dependence on its own (other than ρ) parameters, which are parameters of a family for the initial equation (2). The possibility of moving the initial and limit points over the complex p-plane by varying the parameters of the family gives us considerable influence on the dynamics of the system and enables us to modify it in the desired direction.

We note, however, that not every parameter of the family exerts an independent influence on the initial and limit points, since it may appear in both of these equations. Some of the initial and limit points are determined by the lumped parameters of the blocks in the system, i.e., by the polynomials Φ_i, Λ, Ψ_i, Θ. There are also infinite numbers of initial and limit points associated with the presence of the delays, i.e., with the exponents $e^{p\tau_i}$. Generally speaking, these points may be anywhere on the complex p-plane, including the region finite in δ, and not only at $+\infty$ and $-\infty$, as in the case of single-loop systems with a single delay.

It can be shown that the form of the root loci is related only to the relative positions of the initial and limit points and does not depend on the position of the imaginary axis. In fact, any term of Eq. (2) can be written [7] in the form of a product $\sigma\prod_{\alpha}(p-p_\alpha)\,e^{p\tau}$, where p_α are the roots of the corresponding polynomial and σ is the parameter of the family. Obviously, the difference $p - p_\alpha$ does not depend on the position of the imaginary axis and is the distance from the position of the root to a certain point on the p-plane. The term $e^{-p\tau}$ that appears on a shift of the imaginary axis through a is always real and can be taken into account in the value of the corresponding family parameter. The values of the family parameters do, of course, change in the new position of the imaginary axis when the relative positions of the initial and limit points have been left unchanged. Thus, the configuration of the root loci described by Eq. (2) does not change on substitution of a variable of the form $\tilde{p}=p+a$, where a is real. Consequently, each given disposition of the initial and limit points and the corresponding root portrait determine a whole continuum of different systems corresponding to different positions of the imaginary axis.

As in systems without delay, it can be shown that any group of roots of (2) with a fixed $\rho = \rho_0$ can be taken as the initial points. The limit points do not change. This property makes it possible to simplify the investigation in some cases.

It is known [7] that root loci can be constructed by a geometric method, from the fundamental phase equation, or analytically, from the analytic equation of the root loci. In our case, although it is possible in principle to obtain the fundamental phase equation, many unwieldy trigonometric operations are required. The complexity of the final formula

increases rapidly with increasing number of terms in (2). Moreover, the simple geometric content of the fundamental phase equations in simpler cases [3,7] is lost even for an equation with one delay of general form [4]. Finally, it is more convenient to use the analytic method for programming to compute root loci on a digital computer. It will therefore be used below.

As always, the analytic root-locus equation can be obtained by separating the real and imaginary parts in (2), equating them to zero, and eliminating the free parameter. The equation has the form

$$
\begin{aligned}
F(\delta, \omega) = {} & (\Phi_{1r}\Psi_{1j} - \Phi_{1j}\Psi_{1r})\, e^{2\delta\tau_1} + \ldots + (\Phi_{kr}\Psi_{kj} - \Phi_{kj}\Psi_{kr})\, e^{2\delta\tau_k} + \\
& + [(\Phi_{1r}\Psi_{2j} - \Phi_{1j}\Psi_{2r} + \Phi_{2r}\Psi_{1j} - \Phi_{2j}\Psi_{1r}) \cos\omega(\tau_1 - \tau_2) - \\
& - (\Phi_{1r}\Psi_{2r} + \Phi_{1j}\Psi_{2j} - \Phi_{2r}\Psi_{1r} - \Phi_{2j}\Psi_{1j}) \sin\omega(\tau_1 - \tau_2)]\, e^{\delta(\tau_1+\tau_2)} + \ldots \\
\ldots + {} & [(\Phi_{k-1,r}\Psi_{kj} - \Phi_{k-1,j}\Psi_{kr} + \Phi_{kr}\Psi_{k-1,j} - \Phi_{kj}\Psi_{k-1,r}) \cos\omega(\tau_{k-1} - \tau_k) - \\
& - (\Phi_{k-1,r}\Psi_{kr} + \Phi_{k-1,j}\Psi_{kj} - \Phi_{kr}\Psi_{k-1,r} - \Phi_{kj}\Psi_{k-1,j}) \times \\
& \times \sin\omega(\tau_{k-1} - \tau_k)]\, e^{\delta(\tau_{k-1}+\tau_k)} + \\
& + [(\Phi_{1r}\Theta_j - \Phi_{1j}\Theta_r + \Lambda_r\Psi_{1j} - \Lambda_j\Psi_{1r}) \cos\omega\tau_1 - \\
& - (\Phi_{1r}\Theta_r + \Phi_{1j}\Theta_j - \Lambda_r\Psi_{1r} - \Lambda_j\Psi_{1j}) \sin\omega\tau_1]\, e^{\delta\tau_1} + \ldots \\
\ldots + {} & [(\Phi_{kr}\Theta_j - \Phi_{kj}\Theta_r + \Lambda_r\Psi_{kj} - \Lambda_j\Psi_{kr}) \cos\omega\tau_k - \\
& - (\Phi_{kr}\Theta_r + \Phi_{kj}\Theta_j - \Lambda_r\Psi_{kr} - \Lambda_j\Psi_{kj}) \sin\omega\tau_k]\, e^{\delta\tau_k} + \Lambda_r\Theta_j - \Lambda_j\Theta_r = 0,
\end{aligned}
\tag{3}
$$

where the subscripts r and j denote the real and imaginary parts of the corresponding polynomials. Despite the complexity of this equation in general form, it can usually be simplified substantially in specific cases owing to the absence of some of the polynomials in the characteristic equation.

The general trends of the root-locus continuations become clear after plotting of the initial and limit points on the complex plane and therefore they need be plotted exactly only for the regions of interest to us. We note also that the general form of Eq. (3) does not depend on the number of delay elements k. Addition of new delays results in a simple increase in the number of terms of this equation. Thus, the fundamental complexity of the analytic root-locus equation ceases to increase beginning at k = 2, i.e., in the presence of two delays in the system. This greatly facilitates programming.

The program written for determination of root paths from the analytic equation (3) on the small Mir-1 digital computer indicated good computing quality coupled with comparatively small machine times. The initial data are written in the form of the coefficients of the polynomials of the characteristic equation (2) and values of the family parameters and delay times, i.e., practically no preliminary manual work on the information is necessary. Since the left-hand side of the analytic equation (3) is a function of the two variables δ and ω, we can, by fixing, for example, δ and moving along a line parallel to the imaginary axis, find all roots of this equation that lie in a certain frequency range. These roots are, of course, also roots of (2). Thus, the root-locus method makes it possible to circumvent in natural fashion the difficulties associated with the infinite number of roots of the transcendental characteristic equation. For each point of the locus that has been found, the program simultaneously computes the value of the free parameter. The parameter formulas are obtained either directly from (2) by calculating the modulus or from the real or imaginary part of (2).

Let us consider certain general properties of the root loci that follow from the initial characteristic equation (2) and the analytic root-locus equation (3). Recognizing that ω appears as a multiplier in the imaginary part of each polynomial [7], it is easily seen that $\omega = 0$ is a solution of this equation and, consequently, that the real axis in its entirety is part of the root loci. Expressing the free parameter ρ from Eq. (2), we find that large δ on the right semiaxis $\rho < 0$ are sufficient, since the coefficients in higher degrees of $\Phi_1(p)$ and $\Psi_1(p)$ can always be reduced to unity. The sign of ρ will change upon passage through each initial or limit point toward the left on the real axis. Thus, the real axis is partitioned by the initial and limit points into segments with $\rho > 0$ and $\rho < 0$ as usual, after which no difficulty is encountered in identification of loci with a given sign of the free parameter over the entire complex plane as well.

If we substitute $\delta = 0$ into (3), we obtain an equation for determination of the frequencies at which the roots of the characteristic equation (2) reach the imaginary axis.

Eliminating the free parameter from (2) and from the equation obtained from it by differentiation with respect to p, we find the equation of the possible multiple points. However, since this transcendental equation will be much more complex than (2), it would hardly be convenient to solve it directly. The approximate locations of multiple points on the real axis can be established from general considerations of root-locus theory. With practice, the same thing can also be done for complex multiple points. Admittedly, multiple points can exist in this case only theoretically, since trajectories of one type actually become trajectories of another type on a small change in the parameters of the family. The necessary adjustments in the positions of the multiple points are made during computation of the root loci.

Knowledge of the asymptotic form gives a general idea of the root portrait of the system, one that can be used to isolate certain regions of the p-plane in which exact plotting of the loci is necessary. Various new and unusual properties appear in the presence of several delays. It will be recalled that for systems without delays, the asymptotes of the root loci were straight lines forming a regular star centered on the real axis [7]. For single-loop systems with one delay, the asymptotes were horizontal straight lines separated by distances π/τ [3], and in the general case of one-delay systems they were certain combinations of these asymptotic forms [4]. The basic types of root-locus asymptotes for (2) are as follows.

As $\delta \to -\infty$ and $\delta \to \infty$, the root loci of Eq. (2) tend to the asymptotes of the equations

$$\Lambda(p) + \rho\Theta(p) = 0, \tag{4}$$

$$\Phi_1(p) + \rho\Psi_1(p) = 0. \tag{5}$$

Equations of this type describe systems without delay, and a star of asymptotes corresponds to each of them. In the first star, only the rays emanated into the left half-plane ($\delta \to -\infty$) need be used, and in the second only those going into the right half-plane ($\delta \to \infty$). Thus, Eq. (2) has two half-stars of asymptotes to which a finite number of the roots of this transcendental equation tend. Here an infinite number of its roots may remain in a region of the p-plane that is finite in δ if all of the initial and limit points are in this region. Consequently, with several delays, as in the general case of one delay [4], it is possible to construct systems whose dynamics are similar to the dynamics of systems without delay. For example, stability can be preserved in the presence of unlimited increase of the free parameter, including the gain. This last possibility was first pointed out by M. V. Meerov [8].

In particular cases in which individual polynomials vanish identically in (2), the asymptotic behavior is as follows. If $\Lambda(p) \equiv 0$ or $\Theta(p) \equiv 0$, the asymptotic properties as $\delta \to \infty$ are determined by Eq. (5). As $\delta \to -\infty$, if $\Lambda(p) \equiv 0$, the asymptotes correspond to the asymptotes of the equation

$$\Phi_k(p)\, e^{p\tau_k} + \rho\Theta(p) = 0, \tag{6}$$

and for $\Theta(p) \equiv 0$ to those of the equation

$$\Lambda(p) + \rho\Psi_k(p)\, e^{p\tau_k} = 0. \tag{7}$$

Since $\delta \to -\infty$, there will be horizontal asymptotes of these equations separated by $\frac{\pi}{\tau_k}$, only in the left half-plane. Although for Eq. (6) the roots move, as usual, from $-\infty$ to the right as the free parameter increases, the picture is reversed for (7): the roots move from right to left to $-\infty$.

In the case $\Phi_1(p) \equiv 0$ or $\Psi_1(p) \equiv 0$ as $\delta \to -\infty$, the asymptotic behavior is determined by (4). For $\delta \to \infty$, it will be determined at $\Phi_1(p) \equiv 0$ by the equation

$$\Phi_2(p) + \rho\Psi_1(p)\, e^{p(\tau_1 - \tau_2)} = 0, \tag{8}$$

and at $\Psi_1(p) \equiv 0$ by

$$\Phi_1(p)\, e^{p(\tau_1 - \tau_2)} + \rho\Psi_2(p) = 0. \tag{9}$$

The asymptotes of these equations are also horizontal lines, but they are a greater distance apart: $\frac{\pi}{\tau_1 - \tau_2}$. Since $\delta \to \infty$ these asymptotes lie in the right half-plane. As the

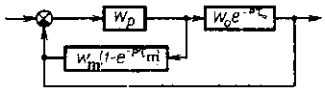

Fig. 1

free parameter increases for Eq. (8), the roots move from right (from $+\infty$) to left, and for (9) from left to right (to $+\infty$).

Identical vanishing of the other polynomials results either in similar cases or combinations of similar cases.

As we have noted, the initial and limit points can be shifted by varying the parameters of the family, e.g., they can all be placed to the left of the imaginary axis. Introducing correcting elements and couplings, we can change the asymptotic form and make certain roots approach the imaginary axis. All of this makes it possible to synthesize systems with predetermined properties.

To illustrate application of the general theory, let us consider the so-called Smith's linear leader [9]. Its block diagram appears in Fig. 1. Here the main control loop consists of a controller with a transfer function W_p and the controlled object, which contains the block W_0 and a pure delay $e^{-p\tau 0}$. A correcting feedback in which W_0 and $e^{-p\tau 0}$ are modelled by the functions W_m and $e^{-p\tau m}$ is introduced to compensate the delay. The closed-system transfer function has the form

$$W = \frac{W_p W_0 e^{-p\tau_0}}{1 + W_p W_m + W_p [W_0 e^{-p\tau_0} - W_m e^{-p\tau_m}]}.$$

In accurate modelling, the expression in the square brackets equals zero, and the terms containing exponentials are eliminated from the characteristic equation. Thus, the free motion of this system will correspond to the motions of a system without delay. However, since exact coincidence of the above quantities is impossible, it is logical to regard this system as containing two delays.

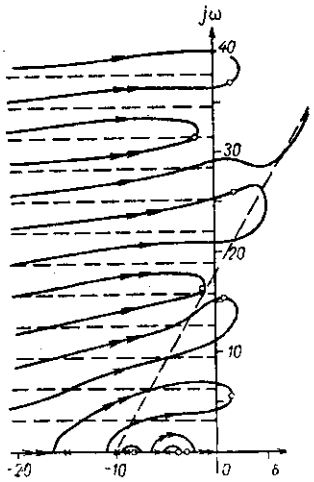

Fig. 2

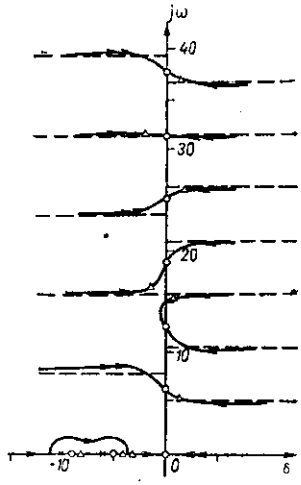

Fig. 3

Let the transfer functions of the respective blocks have the forms

$$W_p = \frac{K_p}{p + 15}, \quad W_0 = \frac{K_0}{(p+5)(p+9)}, \quad W_m = \frac{K_m}{(p+6)(p+10)},$$

and let the delay times be $\tau_m = 1$ and $\tau_0 = 0.6$. Then the characteristic equation (2) assumes the form

$$\Phi(p) e^{p\tau_m} + \rho [\sigma \Psi_1(p) e^{p\tau_m} + \Psi_2(p) e^{p\Delta\tau} + \sigma\Theta(p)] = 0, \tag{10}$$

where

$$\rho = K_p K_0, \ \sigma = \frac{K_m}{K_0}, \ \Delta\tau = \tau_m - \tau_0 = 0{,}4,$$

or, substituting the numerical values:

$$(p+5)(p+6)(p+9)(p+10)(p+15)e^{p}+ \\ +\rho[\sigma(p+5)(p+9)(e^{p}-1)+(p+6)(p+10)e^{0.4p}]=0. \tag{11}$$

It follows from the above that the root-locus asymptotes of Eq. (10) are horizontal straight lines separated by a distance $\frac{\pi}{\tau_m}$ on the left and inclined straight lines on the right. Since the difference between the degrees of the polynomials $\Phi(p)$ and $\Psi_1(p)$ is three, two asymptotes extend to the right at angles of ±60° to the real axis. The root loci of (11) are shown in Fig. 2 for the family parameter $\sigma = 0.5$. Here the single arrowheads mark trajectories on which $\rho > 0$, while the double ones indicate $\rho < 0$. The crosses indicate the initial points and the open circles the limit points. At $\rho_{cr} = 1147$, a pair of roots moves out onto the right half-plane at the frequency $\omega_{cr} = 4.18$. The roots corresponding to higher frequencies are still to the left of the imaginary axis at this value of the free parameter. We noted above that shifting of the initial and limit points on the p-plane on a change in the family parameters has a significant effect on system dynamics. It is seen from Eq. (10) and Fig. 2 that the initial points are at $-\infty$ and that the limit points can be moved by varying the family parameter σ ($\sigma = 1$ when the model and the controlled object have exactly equal gains). The root locai for the limit points are shown in Fig. 3 as functions of σ. Here the asymptotes are horizontal lines, but the distance between them is $\frac{\pi}{\Delta\tau}$, in the left half-plane and $\frac{\pi}{\tau_m-\Delta\tau}=\frac{\pi}{\tau_0}$ on the right. As σ increases, the roots move toward one another from $-\infty$ and $+\infty$. The triangles indicate the positions of the points at $\sigma = 0.5$, which corresponds to the limit-point positions in Fig. 2.

It is evident from Fig. 3 that some of the limit points are far to the left of the imaginary axis at small σ. This means that the roots of the characteristic equation that move toward these limit points quickly enter the right half-plane, and the system loses stability even at small values of the free parameter ρ, i.e., of the gains in the forward channel (Fig. 2). An increase in σ moves the limit points closer to the imaginary axis, but some of them remain in the right half-plane in principle. It is now clear that if the exact equality $\tau_m = \tau_0$ is not observed, then $\sigma = 1$, i.e., equality of the model and object gains is not the optimum. The accuracy of modelling of the W_m links is of no particular importance, as the figure shows, since the difference between W_m and W_0 introduces only local changes into the root-locus picture.

The theory set forth above and the example illustrating it indicate the possibilities of the root-locus method for analysis of complex systems with delay.

REFERENCES

1. M. A. Borovnikov, A. P. Ineshin, and G. V. Loginov, Materials of the First Volga Conference on Automatic Control [in Russian], Kazan, book 1, 1971.
2. Huang Bau, H. Walter, and Ken Sheu Lu Chen, Int. J. Control, vol. 18, no. 3, 1973.
3. G. A. Bendrikov and F. B. Konev, Avtomatika i telemekhanika, no. 4, 1969.
4. G. A. Bendrikov and V. V. Volkov, Vestn. Mosk. un-ta. Fizika [Moscow University Physics Bulletin], vol. 15, no. 4, 1974.
5. L. E. El'sgol'ts and S. B. Norkin, Introduction to the Theory of Differential Equations with Deviating Argument [in Russian], Moscow, 1971.
6. L. S. Pontryagin, Izv. AN SSSR. Matem., vol. 6, no. 3, 1942.
7. G. A. Bendrikov and K. F. Teodorchik, Root Loci of Linear Automatic Control Systems [in Russian], Moscow, 1964.
8. M. V. Meerov, Structural Synthesis of High-Precision Automatic Control Systems [in Russian], Moscow, 1967.
9. P. S. Buckley, Proceedings of First International Conference on Automatic Control [in Russian], Moscow, vol. 1, 1961.

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