ELECTRON BREMSSTRAHLUNG ON NUCLEI IN STRONG MAGNETIC FIELDS

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Electron bremsstrahlung on nuclei in a strong constant and homogeneous magnetic field is investigated. A formula is obtained for the spectral and angular distribution of the radiation with consideration of its polarization.

The problem of recognizing the nature of radio and optical emission arriving from objects in space has recently been receiving a great deal of attention. It can now be regarded as definite that strong magnetic fields that govern intense (synchrotron) charged-particle emission are not the least important factor here [1]. At fields of the order of 10^9 G and higher, electron bremsstrahlung on nuclei, whose polarization properties are determined by the presence of magnetic fields, may make a substantial contribution. Such fields may exist at the surfaces of neutron stars (pulsars) [2], whose radio emission may perhaps also be due to these effects in the final analysis. This was pointed out by Canuto et al. in [3], where they investigated a case of "one-dimensional" radiation in which the momentum of the electron varies along the field and changes in the dimensions of its "orbit" are left out of account. This approach is justified at nonrelativistic energies in large fields. In calculating the matrix elements, however, these authors used Green's function of the free electron field, which is equivalent to considering the principal term of the expansion in the parameter $\mathcal{H}/\mathcal{H}_0$, where $\mathcal{H}_0 = m_0^2c^3/e_0\hbar$, and, consequently, their method cannot yield a satisfactory result at fields $\mathcal{H}$ that are comparable with the critical field $\mathcal{H}_0$.

In our approach, we divide by the Dirac operator [4,5]:

$$\hat{D} = i\hbar\frac{\partial}{\partial t} - c(\boldsymbol{\alpha}\mathbf{P}) - \rho_3 m_0 c^2, \tag{1}$$

which makes it possible to avoid such limitations on the magnitude of the magnetic field (here P is the generalized momentum operator and the other nomenclature is conventional).

The probability of transition of an electron in a unit time from state ψ_i to the final state ψ_f is given by

$$w = \frac{\partial}{\partial t}(C_{if}^{+}C_{if})|_{t\to\infty}, \tag{2}$$

where

$$C_{if} = -\frac{i}{\hbar}\int_0^t dt' \int d^3x \psi_f^{+} W^{+} \psi_i,$$

and the operator W^+ has the form

$$W^{+} = \frac{4\pi Z e_0^3}{L^{9/2}} \sum_{\varkappa_t,\varkappa'} \sqrt{\frac{2\pi c\hbar}{\varkappa}} \cdot \frac{e^{i\varkappa' R}}{\varkappa'^2} \{(\boldsymbol{\alpha}\mathbf{a}^{+}) e^{ic\varkappa t - i\varkappa r} \times \tag{3}$$

$$\times D^{-1} e^{-i\varkappa' r} + e^{-i\varkappa' r} D^{-1} (\boldsymbol{\alpha}\mathbf{a}^{+}) e^{ic\varkappa t - i\varkappa r}\}.$$

Here the $\mathbf{a}^+$ are the operators for production of a photon with momentum $\varkappa$ and a given polarization, Ze_0 is the charge of the nucleus at rest, and R is its radius vector. The operation of division by D is so defined as to satisfy the relation $DD^{-1} = 1$. Using the Hermitian property of the operator D^{-1}, we obtain

$$w(n',\ s',\ k_3') = \frac{16\pi^2\alpha^2Z^2c}{L^5}\sum_{\varkappa}\frac{K_0^2(|\varkappa_3'|R_\perp)}{\varkappa}|u_{if}|^2\delta(K-K'-\varkappa), \tag{4}$$

where

$$u_{if} = \int dxdye^{-i(\varkappa r)_\perp}\psi^+_{n's'}\cdot\left[(\alpha a+)\frac{2K-\alpha_3\varkappa_3'}{\varkappa_3'(2k_3-\varkappa_3')} - \frac{2K'+\alpha_3\varkappa_3'}{\varkappa_3'(2k_3'+\varkappa_3')}(\alpha a+)\right]\psi_{ns}.$$

The z axis coincides with the direction of the constant and homogeneous magnetic field $\mathcal{H}$, and

$$\psi = \frac{1}{\sqrt{L}}\psi_{ns}e^{-icKt+ik_3z}$$

is the wave function of the electron in this field, which describes a state with energy $E=c\hbar K$, a z-component of momentum $p_3=\hbar k_3$, the principal quantum number n and the radial number s, with $K=\sqrt{4\gamma n+k_3^2+k_0^2}$ (the solution in cylindrical coordinates is used [4]). The primed quantities pertain to the final state of the electron. We have also introduced the following notation:

$$\alpha = e_0^2/c\hbar,\quad R_\perp = \sqrt{R_1^2+R_2^2},\quad (\varkappa r)_\perp = \varkappa_1 x+\varkappa_2 y,$$
$$k_0 = m_0c/\hbar,\quad \gamma = e_0\mathcal{H}/2c\hbar,\quad \varkappa_3' = k_3-k_3'-\varkappa_3.$$

We note that formula (4) was obtained using the approximate equality $\exp\{-i(\varkappa' r)_\perp\}\approx 1$, which follows from the condition

$$(\bar{r}/R_\perp)\ll 1,\quad \bar{r}\sim(n/\gamma)^{1/2}, \tag{5}$$

since terms with $\varkappa_{1,2}'\leqslant R_\perp^{-1}$ make the main contribution to the sum over the $\varkappa'$ in (3). Condition (5) is equivalent to the following relation linking the possible values of $R_\perp$ that we are considering to the electron energy E and the field strength $\mathcal{H}$:

$$R_\perp \gg \frac{1}{k_0}\frac{k_\perp}{k_0}\frac{H_0}{H},\qquad k_\perp = \sqrt{4\gamma n}. \tag{6}$$

Given satisfaction of (5), the summation over $\varkappa'$ in (3) is carried out by the formula

$$\frac{1}{L^2}\sum\frac{e^{i\varkappa' R}}{\varkappa'^2}\to\frac{e^{i\varkappa_3'R_3}}{(2\pi)^2}\int\frac{e^{i(\varkappa' R)_\perp}}{\varkappa'^2}d\varkappa_1'd\varkappa_2' = \frac{e^{i\varkappa_3'R_3}}{2\pi}K_0(|\varkappa_3'|R_\perp),$$

where $K_0(x)$ is a Macdonald function.

The matrix elements in (4) are calculated from the known formulas of [4], and, after averaging over the initial spin states of the electron and summing over the final states, we obtain for the π-component of the radiation

$$\begin{aligned} dw_\pi(n',\ \varkappa,\ \theta) = {} & \frac{2c\alpha^2Z^2}{\pi L}\frac{\varkappa K'}{|k_3'|}\sum\frac{K_0^2(|\varkappa_3'|R_\perp)}{\varkappa_3'^2(2k_3-\varkappa_3')^2(2k_3'+\varkappa_3')^2}\times \\ & \times\left\{q^2\left[\left(1-\frac{k_0^2+k_3k_3'}{KK'}\right)(I^2_{n,n'-1}+I^2_{n-1,n'}) - 2\beta_\perp\beta_\perp' I_{n,n'-1}I_{n-1,n'}\right]+\right. \\ & +\varkappa_3^2\varkappa_3'^2\left[\left(1+\frac{k_0^2-k_3k_3'}{KK'}\right)(I^2_{n,n'-1}+I^2_{n-1,n'}) + 2\beta_\perp\beta_\perp' I_{n,n'-1}I_{n-1,n'}\right]- \\ & \left.-2q\varkappa_3\varkappa_3'\left(\frac{k_3'}{K'}-\frac{k_3}{K}\right)(I^2_{n,n'-1}+I^2_{n-1,n'})\right\}d\varkappa\sin\theta d\theta, \end{aligned} \tag{7}$$

where $q=K(2k_3'+\varkappa_3')-K'(2k_3-\varkappa_3')$, $x=\frac{\varkappa^2\sin^2\theta}{4\gamma}$, θ is the angle between $\varkappa$ and H, and the sum is taken over the two possible values $k_3'=\pm\sqrt{(K-\varkappa)^2-4\gamma n'-k_0^2}$. The function $I_{n,m}(x)$ in (7) is related to the Laguerre polynomials

$$I_{n,m}(x) = \frac{e^{-\frac{x}{2}}}{\sqrt{n!m!}}x^{\frac{n-m}{2}}Q_m^{n-m}(x),$$

and the summation over s' was carried out by the formula $\sum_{s'} I^2_{ss'}(x) = 1$. Similarly, we obtain for the σ-component of the radiation

$$dw_\sigma(n', \varkappa, \theta) = \frac{2c\alpha^3 Z^2}{\pi L} \frac{\varkappa K'}{|k_3'|} \sum \frac{K_0^2(|\varkappa_3'| R_\perp)}{\varkappa_3'^2 (2k_3 - \varkappa_3')^2 (2k_3' + \varkappa_3')^2} \times \tag{8}$$
$$\times \{A\cos^2\theta + B\sin^2\theta + 2C\sin\theta\cos\theta\} d\varkappa \sin\theta d\theta,$$

where the coefficients A, B, and C have the form

$$A = q^2 \left[\left(1 - \frac{k_0^2 + k_3 k_3'}{KK'}\right)(I^2_{n,n'-1} + I^2_{n-1,n'}) + 2\beta_\perp \beta_\perp' I_{n,n'-1} I_{n-1,n'} \right] +$$
$$+ \varkappa_3^2 \varkappa_3'^2 \left[\left(1 + \frac{k_0^2 - k_3 k_3'}{KK'}\right)(I^2_{n,n'-1} + I^2_{n-1,n'}) - 2\beta_\perp \beta_\perp' I_{n,n'-1} I_{n-1,n'} \right] -$$
$$- 2q\varkappa_3 \varkappa_3' \left(\frac{k_3'}{K'} - \frac{k_3}{K}\right)(I^2_{n,n'-1} + I^2_{n-1,n'}),$$
$$B = q^2 \left[\left(1 - \frac{k_0^2 - k_3 k_3'}{KK'}\right)(I^2_{n-1,n'-1} + I^2_{n,n'}) - 2\beta_\perp \beta_\perp' I_{n-1,n'-1} I_{n,n'} \right] +$$
$$+ \varkappa_3'^2 (k_3 + k_3')^2 \left[\left(1 + \frac{k_0^2 + k_3 k_3'}{KK'}\right)(I^2_{n-1,n'-1} + I^2_{n,n'}) + 2\beta_\perp \beta_\perp' I_{n-1,n'-1} I_{n,n'} \right] -$$
$$- 2q\varkappa_3' (k_3 + k_3') \left(\frac{k_3'}{K'} + \frac{k_3}{K}\right)(I^2_{n-1,n'-1} + I^2_{n,n'}),$$
$$C = \beta_\perp (I_{n,n'-1} I_{n-1,n'-1} + I_{n-1,n'} I_{n,n'}) \left[-q^2 \frac{k_3'}{K'} - \varkappa_3 \varkappa_3'^2 (k_3 + k_3') \frac{k_3'}{K'} + \right.$$
$$\left. + q\varkappa_3' (k_3 + k_3' + \varkappa_3) \right] + \beta_\perp' (I_{n,n'-1} I_{nn'} + I_{n-1,n'} I_{n-1,n'-1}) \left[-q^2 \frac{k_3}{K} + \right.$$
$$\left. + \varkappa_3 \varkappa_3'^2 (k_3 + k_3') \frac{k_3}{K} + q\varkappa_3' (k_3 + k_3' - \varkappa_3) \right].$$

Formulas (8) and (7) describe bremsstrahlung, which generally takes place with a change in the quantum number n (n' ≠ n), i.e., with a change in the radius of the electron's "orbit."

At nonrelativistic electron energies and with satisfaction of the condition $\frac{k_\perp}{k_0} \frac{\mathcal{H}_0}{\mathcal{H}} \left(\frac{k_3}{k_0}\right)^2 \ll 1$, the main contribution will come from transitions with n' = n, as is easily verified by using the asymptotic expansion of the functions $I_{n,m}(x)$, which is valid for $nx \ll 1$:

$$I_{n,m}(x) \approx \delta_{n,m} + \sqrt{nx}\,\delta_{n-1,m} - \sqrt{(n+1)}\,x\,\delta_{n+1,m} + \ldots \tag{9}$$

In the nonrelativistic limit, the π-component of the radiation vanishes and we obtain for the σ-component

$$dw_\sigma^{(n,n)}(\varkappa, \theta) = \frac{8c\alpha^3 Z^2}{\pi L} \cdot \frac{\varkappa k_0}{(k_3^2 - 2k_0\varkappa)^{1/2}} \cdot \left\{ \frac{K_0^2(R_\perp |k_3 - \sqrt{k_3^2 - 2k_0\varkappa}|)}{(k_3 + \sqrt{k_3^2 - 2k_0\varkappa})^2} + \right.$$
$$\left. + \frac{K_0^2(R_\perp |k_3 + \sqrt{k_3^2 - 2k_0\varkappa}|)}{(k_3 - \sqrt{k_3^2 - 2k_0\varkappa})^2} \right\} d\varkappa \sin^3\theta d\theta. \tag{10}$$

If $\mathcal{H} \sim \mathcal{H}_0$ and, consequently, n = 0 in the nonrelativistic case, the result is half as large.

The loss of energy by the electron to radiation per unit time on passage through a substance with N_0 nuclei per unit volume can be determined as follows. In (3), it is necessary to sum over the radius vectors of all nuclei that satisfy the condition (6). Then to find the spectral intensity of the radiation, relations (7), (8), and (10) must be multiplied by the energy of one photon $ch\varkappa$, followed by the substitution

$$\frac{1}{L} K_0^2(|\varkappa_3'| R_\perp) \to F = \frac{1}{L} \left\langle \sum_{\alpha,\alpha'=1}^{N} K_0(|\varkappa_3'| R_\perp^\alpha) K_0(|\varkappa_3'| R_\perp^{\alpha'}) e^{i\varkappa_3'(R_3^\alpha - R_3^{\alpha'})} \right\rangle. \tag{11}$$

where N is the total number of nuclei and the symbol $< >$ stands for the average over the statistical ensemble. We have, obviously,

$$F = \frac{N(N-1)}{L} \langle K_0(|\varkappa_3'|R_\perp^{(1)}) K_0(|\varkappa_3'|R_\perp^{(2)}) \cos[\varkappa_3'(R_3^{(1)} - R_3^{(2)})] \rangle + \\ + \frac{N}{L} \langle K_0^2(|\varkappa_3'|R_\perp) \rangle.$$

The first term can be calculated if the pair correlation function of the ensemble is known. But since it is usually much smaller than the second term,

$$F \approx \frac{N}{L} \int_{\sim\bar{r}}^{\infty} K_0^2(|\varkappa_3'|r) \frac{2\pi r}{L^2} dr = \frac{2\pi N_0}{\varkappa_3'^2} \int_{\sim|\varkappa_3'|\bar{r}}^{\infty} K_0^2(x)\, x dx.$$

In the nonrelativistic case, the lower limit of integration $|\varkappa_3'|\bar{r} \sim \frac{k_\perp}{k_0} \frac{\mathcal{H}_0}{\mathcal{H}} \frac{k_3}{k_0} \ll 1$, i.e., it can be put equal to zero, and there is little change in the integral. Thus, the nuclei whose coordinates do not satisfy condition (6) will make a small contribution to the total radiation (the same conclusion can be reached by recognizing that the intensity of the radiation increases logarithmically with decreasing $R_\perp$, while the number of nuclei decreases linearly). Finally, we have in the nonrelativistic approximation

$$F \approx \frac{\pi N_0}{\varkappa_3'^2}. \tag{12}$$

Substituting this into (10) and integrating over the angle θ, we find for the spectral intensity of the radiation

$$\frac{dI}{d\omega} = \frac{16}{3} \frac{N_0 Z^2 e_0^6}{m_0 c^2} (c^2 p^2 - 2m_0 c^2 \cdot \hbar\omega)^{-1/2}. \tag{13}$$

Integrating over ω, we obtain the total intensity

$$I = \frac{16}{3} N_0 Z^2 e_0^6 \cdot \frac{v}{\hbar m_0 c^3}. \tag{14}$$

As $\hbar \to 0$, $dI/d\omega$ does not, of course, depend on the frequency either. Thus, formula (13) is actually of classical nature. In fact, an elementary classical calculation ($dI/d\omega$) for the case of one-dimensional motion of the charge along a straight line gives exactly the result (13) for the contribution of dipole radiation at low frequencies at $\hbar = 0$ (remembering that (13) must be reduced by half, since $p_\perp = 0$ and n = 0) in one-dimensional motion. The total intensity is finite because the quantum properties of light were taken into account ($\hbar \neq 0$).

The simple result (13) is hopeful for the possibility of observing "one-dimensional" bremsstrahlung in the emission spectra of pulsars.

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