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Light Propagation in a Strong External Magnetic Field with Regard to Quantum Corrections

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Abstract—Linearized motion equations for electromagnetic wave field in the external magnetic field have been solved for the PVLAS experiment configuration within the low-energy approximation of quantum electrodynamics. It has been shown that the wave propagation velocity depends on the initial direction of the plane of wave polarization. Dispersion laws corresponding to the waves with mutually perpendicular directions of polarization have been established. The dependence of ellipticity of laser radiation field on initial polarization and the value of the external magnetic field has been obtained. The ellipticity parameter for the configuration of the system used in the PVLAS experiment has been found.

Key words: quantum electrodynamics.

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INTRODUCTION

In this study, the propagation of an electromagnetic wave in a strong external magnetic field is considered. In a number of experiments carried out by the PVLAS (Polarizzazione del Vuoto con LASer) collaboration [1, 2], the effect of the strong external constant magnetic field directed perpendicular to the axis of propagation of a laser beam on the change in polarization of this beam was studied. First, the collaboration members based their explanation of the obtained results on a hypothesis of the existence of axion fields in the framework of the "dark matter" paradigm. Also, it was stated that the standard quantum-electrodynamic corrections are insufficient for explanation of the results; however, later this statement was recognized as premature [2]. Therefore, it is of interest to calculate this effect with regard to the quantum-electrodynamic corrections under the conditions of this experiment. As is known, in the low-energy limit two additional terms occur in the effective quantum-electrodynamics Lagrangian: an Uehling Lagrangian [3] and an Euler–Heisenberg Lagrangian [4] (see also [5, 6]). The total Lagrangian yields nonlinear equations that are reasonable to be linearized by the laser radiation field. It should be noted that the electrodynamics Lagrangian with the Uehling and Euler–Heisenberg additions is widely used, in particular, for studying nonlinear electrodynamics effects in strong fields created, for example, by pulsars [7].

2. LAGRANGIAN. MOTION EQUATIONS

The starting point of the consideration is the low-energy effective Lagrangian, which represents a sum of the classical Lagrangian of electromagnetic field and the Lagrangians corresponding to loop quantum corrections, i.e., the Uehling Lagrangian (photon eigenenergy) and the Euler–Heisenberg Lagrangian (light-by-light scattering). These Lagrangians can be found in [6]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{C_u}{m^2}F_{\mu\nu}\Box F^{\mu\nu} + \left[\frac{\alpha^2}{90m^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right], \quad (1)$$

where the electromagnetic field tensor is

$$F_{\mu\nu} = \partial_\nu A^\mu - \partial_\mu A^\nu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix},$$

$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\rho\rho\sigma}F_{\rho\sigma}$ is the tensor dual to $F^{\mu\nu}$, $\Box = \partial_\nu\partial^\nu$ is the d'Alembertian, $C_u = \frac{\alpha}{60\pi}$, $\alpha = 1/137$ is the fine-structure constant, and m is the electron mass.

Varying the action and putting the variation to zero in accordance with the principle of least action $\delta S = \delta\int \mathcal{L}\, d^4x = 0$, we obtain the nonlinear motion equations:

$$\partial_\nu F^{\mu\nu} - \frac{4C_u}{m^2}\partial_\nu\Box F^{\mu\nu} - \frac{8\alpha^2}{90m^4}\partial_\nu(F_{\rho\sigma}F^{\rho\sigma}F^{\mu\nu}) - \frac{56\alpha^2}{360m^4}\partial_\nu(F_{\rho\sigma}\tilde{F}^{\rho\sigma}\tilde{F}^{\mu\nu}) = 0. \quad (2)$$

As was mentioned above, we will consider the system which represents an electromagnetic wave propagating in the strong external constant magnetic field. In order to study the effect of the external field on the wave characteristics, it is sufficient to consider the equations linear in the radiation field. Thus, we represent the tensor $F^{\mu\nu}$ in the following form:

$$F^{\mu\nu} = B^{\mu\nu} + f^{\mu\nu}, \tag{3}$$

where tensor $B^{\mu\nu}$ describes the strong external magnetic field and tensor $f^{\mu\nu}$, the laser radiation field. The motion equations (2) take the form:

$$\begin{aligned}
&\partial_\nu B^{\mu\nu} - \frac{4C_u}{m^2}\partial_\nu \Box B^{\mu\nu} - \frac{8\alpha^2}{90m^4}\partial_\nu (B_{\rho\sigma}B^{\rho\sigma}B^{\mu\nu}) \\
&- \frac{56\alpha^2}{360m^4}\partial_\nu (B_{\rho\sigma}\tilde{B}^{\rho\sigma}\tilde{B}^{\mu\nu}) + \partial_\nu f^{\mu\nu} - \frac{4C_u}{m^2}\partial_\nu \Box f^{\mu\nu} \\
&- \frac{8\alpha^2}{90m^4}\partial_\nu (2B_{\rho\sigma}f^{\rho\sigma}B^{\mu\nu} + f_{\rho\sigma}f^{\rho\sigma}B^{\mu\nu} \\
&+ B_{\rho\sigma}B^{\rho\sigma}f^{\mu\nu} + 2B_{\rho\sigma}f^{\rho\sigma}f^{\mu\nu} + f_{\rho\sigma}f^{\rho\sigma}f^{\mu\nu}) \\
&- \frac{56\alpha^2}{360m^4}\partial_\nu (2B_{\rho\sigma}\tilde{f}^{\rho\sigma}\tilde{B}^{\mu\nu} + f_{\rho\sigma}\tilde{f}^{\rho\sigma}\tilde{B}^{\mu\nu} \\
&+ B_{\rho\sigma}\tilde{B}^{\rho\sigma}\tilde{f}^{\mu\nu} + 2B_{\rho\sigma}\tilde{f}^{\rho\sigma}\tilde{f}^{\mu\nu} + f_{\rho\sigma}\tilde{f}^{\rho\sigma}\tilde{f}^{\mu\nu}) = 0.
\end{aligned} \tag{4}$$

It is obvious that the external magnetic field can be assumed classical:

$$\partial_\nu B^{\mu\nu} = 0.$$

Using the Bianchi identity $\partial_\nu \tilde{f}^{\mu\nu} = 0$ and neglecting the terms of the order higher than first in field $f^{\mu\nu}$ we obtain the linear equations for the laser radiation field:

$$\begin{aligned}
&\partial_\nu f^{\mu\nu} - \frac{4C_u}{m^2}\partial_\nu \Box f^{\mu\nu} - \frac{8\alpha^2}{90m^4}\partial (B_{\rho\sigma}B^{\rho\sigma}) f^{\mu\nu} \\
&- \frac{16\alpha^2}{90m^4}\partial_\nu (B_{\rho\sigma}f^{\rho\sigma})B^{\mu\nu} - \frac{112\alpha^2}{360m^4}\partial_\nu (B_{\rho\sigma}\tilde{f}^{\rho\sigma})\tilde{B}^{\mu\nu} = 0.
\end{aligned} \tag{5}$$

In order to check this result, representation (3) for tensor $F^{\mu\nu}$ was substituted into initial Lagrangian (1) and the terms that were quadratic in field $f^{\mu\nu}$ were held. The motion equations obtained using such a second-variation Lagrangian completely coincided with Equations (5), which additionally confirms the correctness of our calculations. Thus, we obtained the linear motion equation for a laser radiation field $f^{\mu\nu}$.

3. DISPERSION EQUATIONS

Let us chose a coordinate system where the external field $\mathbf{B} = B\mathbf{j}$ is directed along the y axis and laser radiation propagates along the z axis. Then,

$$f^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & H_y \\ E_y & 0 & 0 & -H_x \\ 0 & -H_y & H_x & 0 \end{pmatrix}, \quad B^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \end{pmatrix},$$

$$\tilde{f}^{\mu\nu} = \begin{pmatrix} 0 & -H_x & -H_y & 0 \\ H_x & 0 & 0 & -E_y \\ H_y & 0 & 0 & E_x \\ 0 & E_y & -E_x & 0 \end{pmatrix}, \quad \tilde{B}^{\mu\nu} = \begin{pmatrix} 0 & 0 & -B & 0 \\ 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is reasonable to suppose that the strong magnetic field introduces in this space "anisotropy" related to the selected direction along the y axis. It should be noted that the analogous effects take place in the optical phenomena of birefringerence or the Faraday effect. In view of this, we represent the laser radiation field as two linearly polarized waves with planes of polarization parallel to the x and y axes, different wave vectors k_x and k_y, and equal frequencies $\omega_1 = \omega_2 = \omega$:

$$\begin{aligned}
E_x(t) &= E_x \exp(ik_x z - i\omega t), \\
H_y(t) &= H_y \exp(ik_x z - i\omega t), \\
E_y(t) &= E_x \exp(ik_y z - i\omega t), \\
H_x(t) &= H_x \exp(ik_y z - i\omega t).
\end{aligned} \tag{6}$$

Using the Maxwell equations $\mathrm{rot}\mathbf{E} = -\dfrac{\partial \mathbf{H}}{\partial t}$, which follows from the $F^{\mu\nu}$ tensor definition, we obtain: $H_y = \dfrac{k_x}{w}E_x$ and $H_x = -\dfrac{k_y}{w}E_y$.

As

$$B^{\mu\nu}B_{\mu\nu} = 2B^2, \quad B_{\mu\nu}f^{\mu\nu} = 2BH_y,$$
$$B_{\mu\nu}\tilde{f}^{\mu\nu} = -2BE_y,$$

the terms of the form

$$\partial_\nu (B_{\rho\sigma}f^{\rho\sigma})B^{\mu\nu} = \begin{pmatrix} 0 & 2B^2\dfrac{\partial H_y}{\partial z} & 0 & 0 \end{pmatrix},$$

$$\partial_\nu (B_{\rho\sigma}\tilde{f}^{\rho\sigma})\tilde{B}^{\mu\nu} = \begin{pmatrix} 0 & 0 & -2B^2\dfrac{\partial E_y}{\partial t} & 0 \end{pmatrix}$$

contribute only to the components $\mu = 1$ and $\mu = 2$ from Equation (5), respectively. The rest components of this equation are identically equal to zero.

For convenience, we make the following notation:

$$A_1 = \left(1 - \frac{16\alpha^2}{90m^4}B^2\right), \quad A_2 = \frac{4C_u}{m^2},$$

$$A_3 = \frac{32\alpha^2}{90m^4}B^2, \text{ and } A_4 = \frac{224\alpha^2}{360m^4}B^2.$$

In Representation (6), Equation (5) uncouples into two nontrivial equations for wave vectors k_1 and k_2 (as we already mentioned, the components $\mu = 0$ and $\mu = 3$ of this equation yield identities).

$$\mu = 1: \; A_1(k_1^2 - \omega^2) - A_2(k_1^2 - \omega^2)^2 - A_3k_1^2 = 0,$$

$$\mu = 2: \; A_1(k_2^2 - \omega^2) - A_2(k_2^2 - \omega^2)^2 - A_4\omega^2 = 0.$$

Thus, we obtained the dispersion equations characterizing the dependence of the wave vectors on the laser radiation frequency for the two electromagnetic waves with perpendicular planes of polarization that propagate in the strong external magnetic field.

Making the changes $x_1 = k_1^2 - \omega^2$ and $x_2 = k_2^2 - \omega^2$, we obtain the quadratic equations

$$A_2x_1^2 - (A_1 - A_3)x_1 - A_3\omega^2 = 0$$

$$\text{and } A_2x_2^2 - A_1x_2 - A_4\omega^2 = 0,$$

whose solutions are

$$x_1 = \frac{A_1 - A_3 \pm \sqrt{(A_1 - A_3)^2 + 4A_2A_3\omega^2}}{2A_2}, \qquad x_2 = \frac{A_1 \pm \sqrt{A_1^2 + 4A_2A_4\omega^2}}{2A_2}. \tag{7}$$

The solutions corresponding to the "+" sign in Formulas (7) are not physical, as they meet the electromagnetic waves propagating with a velocity that is considerably higher than the speed of light in vacuum. The occurrence of such solutions is not surprising as the effective Lagrangian contains a term with higher derivatives. However, this is not contradictory because Lagrangian (1) meets the low-energy approximation.

Thus,

$$k_1^2 = \omega^2 + \frac{A_1 - A_3 - \sqrt{(A_1 - A_3)^2 + 4A_2A_3\omega^2}}{2A_2}$$

$$\text{and } k_2^2 = \omega^2 + \frac{A_1 - \sqrt{A_1^2 + 4A_2A_4\omega^2}}{2A_2}.$$

Calculate coefficients A_1, A_2, A_3, and A_4 using the data used in the PVLAS experiment [1], i.e., the field strength $B \approx 2.5$ T for the wavelength $\lambda \approx 1000$ nm. We operate in the system of units $\hbar = c = 1$; in these units

$$A_1 \approx 1 - 0.25 \times 10^{-22}, \quad A_2 \approx 0.6 \times 10^3 \text{ GeV}^{-2},$$

$$A_3 \approx 0.5 \times 10^{-22}, \text{ and } A_4 \approx 0.9 \times 10^{-22}.$$

Expanding in small parameters A_3, A_4, and $A_1 - 1$ up to the first-order infinitesimal, we have:

$$k_1^2 = (1 - A_3)\omega^2, \quad k_2^2 = (1 - A_4)\omega^2.$$

Note that the same results would be obtained if we did not use the Uehling term in the initial Lagrangian (1). Coefficient A_2, which follows from the Uehling term, will make a weighty contribution at much higher values of the external magnetic field than those determined by coefficients A_3 and A_4. Such magnetic fields cannot be achieved yet under artificial conditions on Earth. However, these fields can be observed, for example, in pulsars.

Thus, we have two linearly polarized waves propagating with different velocities. In a sense, one can assume that the waves propagate in anisotropic media with different "refractive indices" $\frac{w}{k_1}$ and $\frac{w}{k_2}$. In accordance with the foregoing, if the initial plane of beam polarization is not parallel to the axes, then at the output of the facility we will get an elliptically polarized wave.

4. ELLIPTICITY CALCULATION

Basing on the PVLAS experiment [1, 2], we calculate the change in ellipticity of laser radiation at a distance of 1 meter after the facility input. Assume the input light to be linearly polarized.

Thus,

$$E_x = E_0\cos(\alpha)\cos(k_1z - \omega t),$$

$$E_y = E_0\sin(\alpha)\cos(k_2z - \omega t) \tag{8}$$

$$= E_0\sin(\alpha)\cos(k_1z - \omega t + \delta(z)),$$

where $\delta(z) = (k_2 - k_1)z$. Transforming Equations (8), we obtain the noncanonical ellipse equation:

$$\frac{E_x^2}{E_0^2\cos^2\alpha} + \frac{E_y^2}{E_0^2\sin^2\alpha} - 2\frac{E_xE_y}{E_0^2\sin\alpha\cos\alpha}\cos\delta(z) = \sin^2\delta(z). \tag{9}$$

Diagonalizing the quadratic form (9), we obtain the canonical equation of the ellipse whose semi-axes ratio is named ellipticity ε:

$$\varepsilon = \sqrt{\frac{-\cos\delta(z)\sin\left[\operatorname{arccot}\left(\frac{\cot 2\alpha}{\cos\delta(z)}\right)\right] - \cot 2\alpha\cos\left[\operatorname{arccot}\left(\frac{\cot 2\alpha}{\cos\delta(z)}\right)\right] + \frac{1}{\sin 2\alpha}}{\cos\delta(z)\sin\left[\operatorname{arccot}\left(\frac{\cot 2\alpha}{\cos\delta(z)}\right)\right] + \cot 2\alpha\cos\left[\operatorname{arccot}\left(\frac{\cot 2\alpha}{\cos\delta(z)}\right)\right] + \frac{1}{\sin 2\alpha}}}. \quad (10)$$

At small distances of tens of kilometers, $\delta(z)$ can be considered small. Expanding $\cos\delta(z)$ in small $\delta(z)$ up to the second-order infinitesimal and assuming $\alpha = \frac{\pi}{4}$, we obtain

$$\varepsilon \approx \frac{\delta(z)}{2}.$$

Substituting the one-path length $z = 1$ m [2] to (10), we obtain the ellipticity value

$$\varepsilon \approx 2.1 \times 10^{-16}. \quad (11)$$

We would like to note that in the PVLAS experiment ellipticiy was measured at a characteristic distance of 45 km and, then, the one-path ellipticity change was calculated. The experimental limitation of the one-path ellipticity change is $\varepsilon_{exp} < 3.1 \times 10^{-13}$, which does not contradict the theoretical value (11).

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