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The Properties of a Dynamo Wave in the Case of Intense Meridional Circulation

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Abstract—The solution of a Parker dynamo was constructed for the case of intensive meridional circulation using a method similar to WKB. We show how to build a solution in the transition from the traveling-wave regime to the regime of stationary magnetic field configuration. A solution of the Hamilton–Jacobi equations was discovered for the problem of a dynamo containing a triple point in the complex plane wave vector.

Keywords: Parker dynamo, dynamo wave, meridional circulation, Hamilton–Jacobi equation.

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INTRODUCTION

The generation of the magnetic field of the Sun and stars is usually associated with the dynamo mechanism. The simplest scheme of a dynamo, based on the joint action of differential rotation and the alpha-effect, was proposed by Parker [1].A toroidal magnetic field is obtained from the poloidal under the influence of differential rotation in the convection zone. The inverse process of transformation of the toroidal magnetic field in the poloidal is carried out under the action of the alpha-effect, i.e., as a result of violation of mirror symmetry of convection in a rotating body.

The manifestation of 11-year cyclic solar activity is associated with movement from the poles to the equator of the toroidal magnetic field component, the so-called dynamo wave. In [2] it was shown that the Parker model can be investigated asymptotically. For this research, a method based on the quasi-classical approximation used in quantum mechanics was developed. Note that the method developed in [2] goes back to [3–5], in which a variety of the quasi-classical approximations for the asymptotic study of different modes of generation of large-scale magnetic fields was constructed. In [2] it was possible to construct the wave vector for the dynamo wave and study its behavior. Within the framework of asymptotic analysis the length of the cycle predicted by the Parker model can be estimated. It turned out that the model was one order less than was observed. In [6–9] such difficulties are overcome by accounting for meridional flows of the substance from the equator to the poles. The main idea of the asymptotic investigation method developed in [2] appeared futile for the investigation of a more complex model with meridional circulation. At the same time, the asymptotic investigation of this model shows that it contains regimes significantly different from the Parker scheme without the meridional circulation.

In [2] it was shown that for the simplest Parker scheme only one value of cycle length exists. Accounting for the meridional circulation made it possible to obtain the dependence of the cycle length from the value of the moving velocity of the substance in the convective zone. In [6–9] it was shown that the increase of meridional circulation slows the spread of the dynamo wave. Thus, it is possible to achieve some critical value for the meridional wave, where the dynamo wave transits the stationary growing configuration of the magnetic field. However, in [6–9] it is not shown whether the solution exists in this case and the transition from one regime of behavior of the dynamo wave to another is not described.

The purpose of this work is to show that a solution exists for intensive meridional circulation and to perform the construction of such a solution.

THE MAIN FORMULAS

The equations of a dynamo with meridional circulation have the form:

$$\frac{\partial A}{\partial t} + V(\theta)\frac{\partial A}{\partial \theta} = \alpha B + \frac{\partial^2 A}{\partial \theta^2}, \quad (1)$$

$$\frac{\partial B}{\partial t} + \frac{\partial (V(\theta)B)}{\partial \theta} = D\cos\theta\frac{\partial A}{\partial \theta} + \frac{\partial^2 B}{\partial \theta^2}. \quad (2)$$

Here B is the toroidal magnetic field, A defines the poloidal magnetic field, V is meridional circulation, θ is the latitude, which is measured from the equator, and t is time. The factor $\cos\theta$ corresponds to a decrease in the length of the parallel near the poles. The equations are written in dimensionless variables,

so that the amplitude of the α-effect, the gradient of the corner velocity, and the turbulent diffusion coefficient are combined in the dimensionless dynamo number D. In the diffusion terms the effects of curvature are omitted. For symmetry reasons ($\alpha(-\theta) = -\alpha(\theta)$), Eqs. (1) and (2) can be considered only for one (northern) hemisphere with conditions of antisymmetry (dipole symmetry) or symmetry (quadrupole symmetry) at the equator. The sun's magnetic field has a dipole symmetry, so will we not go beyond it.

Following [2], the asymptotic solution of the system (1), (2) will be sought in the form

$$\begin{pmatrix} \hat{A} \\ \hat{B} \end{pmatrix} = \begin{pmatrix} A \\ \dfrac{B}{|D|^{2/3}} \end{pmatrix}$$

$$= \exp(i|D|^{1/3}S + \gamma t)(\mathbf{f}_0 + |D|^{-1/3}\mathbf{f}_1 + \ldots),$$

where

$$\gamma = |D|^{2/3}\Gamma + |D|^{1/3}\Gamma_1 + \ldots,$$

$$\mathbf{f}_0 = \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \quad \mathbf{f}_1 = \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix}, \ldots,$$

S, μ, and ν are smooth functions and $|D| \gg 1$. This approach is a variation of the WKB method in quantum mechanics [10], so that S as an analog acts as $k = S'$ corresponds to the momentum or wave vector, which in this case is complex. The complex Γ defines an eigenvalue, its real part gives the growth rate, while the imaginary part gives the duration of the activity cycle.

Multipliers $|D|^{2/3}$ in the complex growth rate, and $|D|^{1/3}$ in action are chosen so that the differential rotation, α-effect, eigenvalue, and dissipation are of the same order, and are entered into a higher member of the asymptotic expansion. The meridional circulation is included in the same higher term of the asymptotic solution, at

$$V(\theta) = |D|^{1/3} v(\theta).$$

Substituting the chosen form of the unknown solution in Eqs. (1) and (2), we obtain an algebraic system of equations for μ and ν a solvability condition, which is the dispersion relation for the frequency of the dynamo wave and its wave vector, i.e., the Hamilton–Jacobi equation

$$[\Gamma + ikv + k^2]^2 - i\hat{\alpha}k = 0, \qquad (3)$$

where $\hat{\alpha} = \alpha\cos\theta$. For clarity, we recall the study in (3).

Equation (3) contains the unknowns Γ and k. To find them, we use the fact that Eq. (3) does not contain θ explicitly and we can consider α as a parameter. Then, for a given α and Γ four solutions of Eq. (3) can be represented as points on the complex plane of the momentum k. For changes of the parameter α these points form four branches on the plane k. As for the usual Parker dynamo [2], there is no one branch $k(\alpha)$ that smoothly connects the points corresponding to the pole and the equator. Therefore, we choose Γ so that a smooth solution is made from some two branches; this is the condition for determining Γ. As in the usual Parker dynamo, we believe that combination occurs at the point $\theta = \theta_*$, where α is the maximum. In [6] the validity of this assumption was discussed.

The condition that the two branches of $k(\alpha)$ have a common point is based on addressing the zero derivative of the left side of the Hamilton–Jacobi equations on k. Hence, Γ and k are found from the system

$$H(v(\theta), \Gamma, k, \alpha) = 0, \quad \frac{\partial H(v(\theta), \Gamma, k, \alpha)}{k} = 0. \quad (4)$$

Solving (4), we obtain three branches of Γ and four branches of k. In [8] it was shown that for meridional circulation depending on θ, the configurations of the toroidal magnetic field component are arranged, which depend on which branch of Γ we use.

In [8] it was shown that the dynamo wave becomes a stationary growing configuration if

$$v \geq v_{\text{crit}} = \frac{3}{2}\sqrt[3]{\alpha_*},$$

where $\alpha_* = \alpha(\theta_*)$.

The condition for the existence of the solution (1), (2), satisfying the physical sense of the problem is the smooth joining of the branches of the wave vector k connecting the pole and the equator. For $v < v_{\text{crit}}$, the toroidal magnetic field component exists in the form of a wave propagating from the poles to the equator [8]. At $v > v_{\text{crit}}$, as shown in [8], the existence of a solution depends on the choice of branches of Γ and the latitudinal profile of the meridional circulation. In this case, the existing solution has the form of a permanently increasing configuration of the magnetic field. However, the question of the existence and form of the solution for $v > v_{\text{crit}}$ requires an additional investigation.

2. CONSTRUCTION OF THE SOLUTION FOR $v \geq v_{\text{crit}}$.

Unlike the case $v < v_{\text{crit}}$, where all four branches k are consistently connected with each other, at $v = v_{\text{crit}}$ the structure of the wave vector suffers substantial alterations.

Figure 1a shows the root diagram for k at $v = v_{\text{crit}}$. The points $M_0^{1,2}$, $M_1^{1,2,3}$, $M_0^{3,4}$, and M_1^4 correspond to the ends of four branches. The superscript denotes the number of branches, a double superscript describes the double point (connection point of two branches), and the triple index, a triple point (connection point of three branches). The subscript indicates the value of $\hat{\alpha}/\alpha_*$ in the observed point, i.e., $\hat{\alpha} = 0$ or $\hat{\alpha}/\alpha_* = 1$. As far as we know, the solution of this type (with the

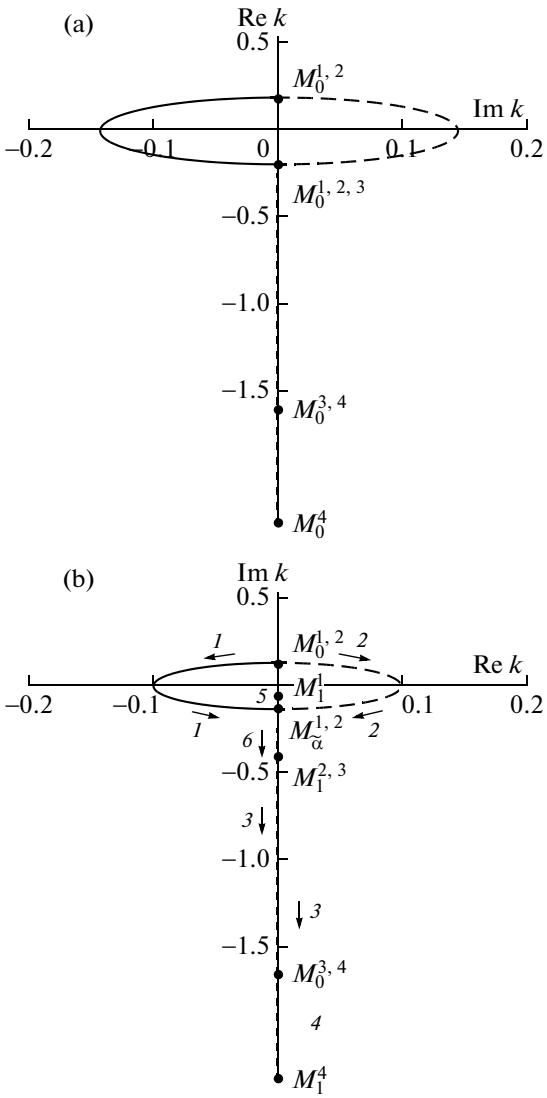

Fig. 1. Four branches of momentum k (a) at $v = v_{crit}$, normalized to $\alpha_*^{1/3}$; (b) at $v > v_{crit}$ and $\Gamma = \Gamma^{(2)}$, normalized to $\alpha_*^{1/3}$.

presence of the triple point) is obtained for the first time.

For $v > v_{crit}$ no smooth solution can exist if the root diagram has a structure similar to Fig. 2b. The conditions under which there is such a structure, described in [8]. Other configurations of the branches of the wave vector, which can be obtained for the dynamo problem with meridional circulation, do not meet the physical sense of the problem and are not considered here. The conditions of their appearance were considered in [8].

In Fig. 1b, the numeration of branches is in correspondence with the decrease of Imk. Recall that the branches $k^{(1)}$ and $k^{(2)}$ are symmetrical about the axis Imk for $\alpha < \tilde{\alpha}$ (or, in other words, $\theta > \tilde{\theta}$, $\tilde{\alpha} = \hat{\alpha}(\tilde{\theta})$).

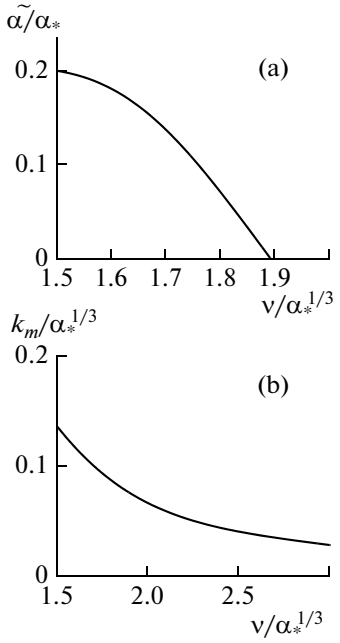

Fig. 2. Dependence of $\tilde{\alpha}/\alpha_*$ from $v/\alpha_*^{1/3}$ (a), of k_m from $v \geq v_{crit}$ (b).

For $\theta_* < \theta < \tilde{\theta}$ (or $\tilde{\alpha} < \hat{\alpha} < \alpha_*$, $\tilde{\alpha} = \hat{\alpha}(\tilde{\theta})$), the roots $k^{(1)}$ and $k^{(2)}$ are purely imaginary. Roots $k^{(1)}$ and $k^{(2)}$ coincide only at $M_{\tilde{\alpha}}^{1,2}$. Figure 3 a shows $\tilde{\alpha}$ depends on v as well.

To construct a solution for the case $v > v_{crit}$, we divide the branches $k^{(1)}$ and $k^{(2)}$ into two sub-branches and at the same time preserve the notation $k^{(1)}$ and $k^{(2)}$ for the two complex subbranches, symmetrical about the axis Imk. For these subbranches $\hat{\alpha} < \tilde{\alpha}$ and $\tilde{\theta} < \theta < \pi/2$. The remaining purely imaginary parts of the branches are denoted by $k^{(5)}$ and $k^{(6)}$ (Im$k^{(5)} >$ Im$k^{(6)}$). Here, $\tilde{\alpha} < \hat{\alpha} < \alpha_*$ and $\theta_* < \theta < \tilde{\theta}$.

Note that the diagrams connect the pole ($M_0^{1,2}$) with the equator ($M_0^{3,4}$). In Fig. 1b the arrows show the contour along which the solution is constructed. The numbers besides the arrows indicate the number of branches and subbranches. We write the solution in the form

$$\begin{pmatrix} A \\ |D|^{2/3} B \end{pmatrix} = C_1 e^{\Gamma^{(i)}|D|^{2/3}t + i|D|^{1/3}S_1} \begin{pmatrix} \mu^{(1)} \\ \nu^{(1)} \end{pmatrix}$$

$$+ C_2 e^{\Gamma^{(i)}|D|^{2/3}t + i|D|^{1/3}S_2} \begin{pmatrix} \mu^{(2)} \\ \nu^{(2)} \end{pmatrix}$$

for $\tilde{\theta} \le \theta \le \pi/2$, where $\hat{\alpha}(\tilde{\theta}) = \tilde{\alpha}$,

$$\begin{pmatrix} A \\ |D|^{2/3} B \end{pmatrix} = C_6 e^{\Gamma^{(i)}|D|^{2/3} t + i|D|^{1/3} S_6} \begin{pmatrix} \mu^{(6)} \\ \nu^{(6)} \end{pmatrix}$$

for $\theta^* \le \theta \le \hat{\theta}$, where $\hat{\alpha}(\theta_*) = \alpha_*$,

$$\begin{pmatrix} A \\ |D|^{2/3} B \end{pmatrix} = C_3 e^{\Gamma^{(i)}|D|^{2/3} t + i|D|^{1/3} S_3} \begin{pmatrix} \mu^{(3)} \\ \nu^{(3)} \end{pmatrix}$$

for $0 < \theta < \theta^*$. Here C_i are constants. One of them can be determined from the normalization of eigenfunctions, while the others are determined from the matching condition $\theta = \theta_*$ and $\theta = \tilde{\theta}$. Superscript $\Gamma^{(i)}$, $\mu^{(i)}$, $\nu^{(i)}$ indicates which branch is used.

θ_m corresponds to the maximum of θ, when $\mathrm{Im}\, k = 0$. We provide an expression for k_m:

$$k_m = \pm\sqrt{\frac{v(\theta_m)^2}{2} - \Gamma^{(i)} - \frac{v(\theta_m)}{2}\sqrt{v(\theta_m)^2 - 4\Gamma^{(i)}}},$$

$$\alpha(\theta_m) = 2(k_m^2 + \Gamma)\, v(\theta_m),$$

where $k_m = k(\theta_m)$. The dependence of k_m for meridional circulation for $\Gamma^{(2)}$ and $v = \mathrm{const}$ is shown in Fig. 2 b.

ASYMPTOTIC EXPANSION OF A HIGHER ORDER

To establish the existence of a solution for the Hamilton–Jacobi equations in the case of $v > v_{\mathrm{crit}}$ it is necessary to show that all branches of the wave vector are matched smoothly. To do this it is necessary to conduct the cross linking of branches at the points $M_1^{2,3}$, and $M_{\tilde{\alpha}}^{1,2}$. For this it is necessary to construct an asymptotic expansion of a higher order. Knowing Γ and k, we can solve the equations for μ and ν, then

$$\mu^{(n)} = (\Gamma^{(i)} + (k^{(n)})^2 + ik^{(n)} v(\theta))\sigma^{(n)},$$
$$\nu^{(n)} = ik^{(n)} \cos\theta\, \sigma^{(n)}, \qquad (5)$$

where $n = 1, 2, 3, 6$ (branches $k^{(4)}$, and $k^{(6)}$ are not used in the construction of the solution), and $\sigma = \sigma^{(n)}$ is a function that can be determined from the asymptotic expansion of a higher order. We leave the second order of asymptotic expansion, which includes μ_1 and ν_1. In [9] it was shown that the condition of solvability of an algebraic system with respect to μ_1 and ν_1 is the equation

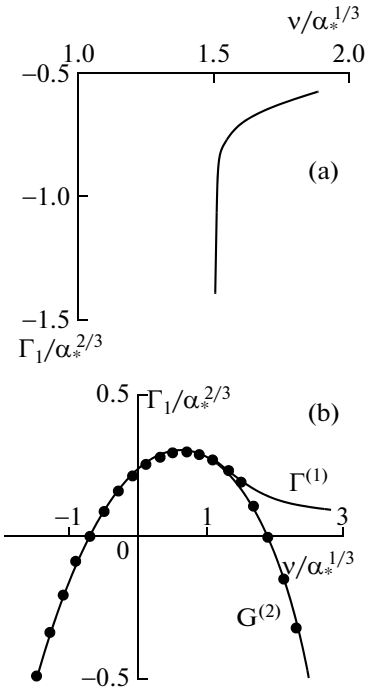

Fig. 3. Dependence of $\Gamma_1/\alpha_*^{2/3}$ from $v/\alpha_*^{1/3}$ (a), of $\Gamma/\alpha_*^{2/3}$ from $v/\alpha_*^{1/3}$ (b).

$$\left[2ik^{(n)} - v(\theta) + \frac{\hat{\alpha}}{2(\Gamma^{(i)} + iv(\theta)k + k^2)}\right](\ln\sigma^{(n)})'$$
$$= \left[\Gamma_1 - i(k^{(n)})'\left(1 + \frac{2(k^{(n)})^2 + 2iv(\theta)k^{(n)} - \frac{v(\theta)^2}{2}}{\Gamma^{(i)} + iv(\theta)k^{(n)} + (k^{(n)})^2}\right)\right. \qquad (6)$$
$$\left. + i\left(k^{(n)} + \frac{iv(\theta)}{2}\right)\tan\theta\right].$$

4. THE SOLUTION NEAR THE POINT $M_1^{2,3}$

We investigate the cross linking of the two branches near the point $M_1^{2,3}$, i.e., how the transition from branch 2 to branch 3 can be performed. We consider (6) near this point. Following [2], we write (6) near the point $\theta = \theta_*$ in the following form:

$$(P_0 + P_1(\theta - \theta_*))(\theta - \theta_*)(\ln\sigma^{(n)})'$$
$$= \Gamma_1 - (Q_0 + Q_1(\theta - \theta_*)).$$

It is seen that the left side of the equation becomes zero at the point $\theta = \theta_*$, σ remains finite at the point

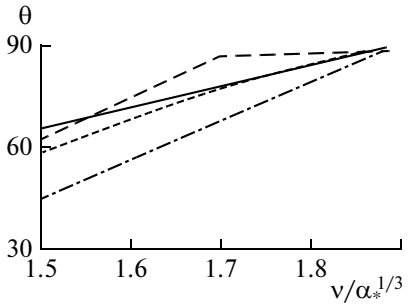

Fig. 4. Dependence of the latitude at which the maximum of the solution is reached from $\nu/\alpha_*^{1/3}$. The asymptotic result θ_m is shown by the solid line, the latitude $\tilde{\theta}$ is shown by the dotted line. The numerical results are shown by dashed lines: the maximum *B* is shown by long dashes, the maximum *A* is shown by short dashes.

θ_*, leading to the vanishing of the right side of the equation. Then, for Γ_1:

$$\Gamma_1 = P_0 n + Q_0.$$

Consequently

$$\Gamma_1 = i(k^{(n)})'\left(3 + \frac{i\nu(\theta_*)}{2k_0^{(n)}}\right)\left(m + \frac{1}{2}\right)$$

$$- i\left(k_0^{(n)} + \frac{1}{2}i\nu(\theta_*)\right)\tan\theta_*,$$

where $n = 2, 3$, $k_0^{(n)} = k^{(n)}(\theta_*)$ and $(k^{(n)})' = (dk^{(n)}/d\theta)|_{\theta=\theta_*}$. In this case, m can take the values 0, 1, 2, 3 .... The expression for the highest eigenfunction ($m = 0$) has the form:

$$\Gamma_1 = -\frac{1}{2}\left[3 + \frac{\nu(\theta_*)}{2\kappa_0^{(n)}}\right]$$

$$\times \frac{\sqrt{2}}{(2\kappa_0^{(n)} + \nu(\theta_*))^{-1} - (\nu(\theta_*) + 2\kappa_0^{(n)})^2} + \kappa_0^{(n)} + \frac{\nu(\theta_*)}{2},$$

where $\kappa_0^{(n)} = i\kappa_0^{(n)}$ and $\kappa_0^{(n)}$ is real and $\kappa_0^{(2)} = \kappa_0^{(3)}$. The graph of Γ_1 is shown in Fig. 3a.

Near the point $M_1^{2,3}$

$$\kappa^{(n)} = \kappa_0^{(n)} - a^{-1}(\alpha - \alpha_*)^{1/2},$$

$$a = -(\nu(\theta_*) + 2\kappa_0^{(n)})^2 + \frac{\alpha_*}{2\kappa_0^{(n)} + \nu(\theta_*)}.$$

However, for $\alpha - \alpha_* \sim (\theta - \theta_*)^2$, the solution is smooth near θ_* and $C_3 = C_6$.

THE SOLUTION NEAR THE POINT $M_{\tilde{\alpha}}^{1,2}$

We use an idea similar to the method of Zwaan (see [11], p. 207, formula (47.4)).

We represent the diffusion equation near the point $M_{\tilde{\alpha}}^{1,2}$, i.e., close to the point $\theta = \tilde{\theta}$ in the following form:

$$\frac{\sigma_1'}{\sigma_1} = -A(\theta - \tilde{\theta})^{1/3} + \ldots,$$

$$\frac{\sigma_2'}{\sigma_1} = A(\theta - \tilde{\theta})^{1/3} + \ldots,$$

here, $\theta > \tilde{\theta}$;

$$\frac{\sigma_5'}{\sigma_1} = iB(\tilde{\theta} - \theta)^{1/3} + \ldots,$$

$$\frac{\sigma_6'}{\sigma_6} = -iB(\tilde{\theta} - \theta)^{1/3} + \ldots,$$

here $\theta < \tilde{\theta}$ and A and B are real and positive. This means that σ exists at the point $M_{\tilde{\alpha}}^{1,2}$, but has no derivative.

We note the essential difference of the equations obtained from (47.7) in [11]. The difference is in the factor before the exponent ($\sim(x-a)^{-1/4}$), whereas we have $\sigma \sim \exp(\tilde{\theta} - \theta)^{1/3}$. This means that our analysis is applicable only in the neighborhood of the point $\tilde{\theta}$, but we can avoid the transition to the complex plane θ, matching the solutions for $\theta > \tilde{\theta}$ and $\theta < \tilde{\theta}$ at the point $\tilde{\theta}$. This leads to the equalities

$$C_6 = C_1 + C_2.$$

The requirement that the decision is real, corresponds to

$$C_1 + C_2.$$

As a result, we have $C_1 = C_2 = C_6/2$, where C_6 is the normalization constant of the eigenfunction considered.

COMPARISON WITH THE NUMERICAL SOLUTION

We compare the results of the WKB approximation with the numerical solution of (1), (2).

Figure 3b shows the values of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ obtained by the WKB method (solid line) and the eigenvalue corresponding to the highest eigenfunction obtained numerically (circles). It is evident that the numerical solution coincides with $\Gamma = \Gamma^{(2)} = \Gamma^{(1)}$ for $\nu/\alpha_*^{1/3} < 3/2$ and coincides with $\Gamma = \Gamma^{(2)}$ for $\nu/\alpha_*^{1/3} > 3/2$.

Figure 4 illustrates the structure of the highest eigenfunction. The solid line shows the location of points θ_m where the WKB solution reaches its maxi-

mum (i.e., Im k = 0) as a function of v. The dashed–dotted line shows the location of the points $\tilde{\theta}$ as a function of v. Recall that $\theta_m > \tilde{\theta}$.

We note that it is impossible to determine the points θ_m directly by numerical simulation (since the same notion of θ_m arises in the WKB method). Instead, the figure shows the point at which B and A are maximal (short dashes and long dashes, respectively). The correspondence between the numerical calculations and analysis is not as ideal for their eigenvalue, but it looks reasonable.

CONCLUSIONS

Increasing the intensity of meridional circulation leads to slower propagation of dynamo waves. When the meridional circulation reaches a certain value, a dynamo wave becomes stationary, increasing the configuration of the magnetic field. This regime is absent in the simplest model of a dynamo. In the present work a solution for this case is shown. Thus, the solution of the Hamilton–Jacobi equation contains a triple point in the complex plane of the wave vector.

From a mathematical point of view, we obtained a method for the transition between different asymptotic regimes. The resulting solution complements and extends the known asymptotes of a class of solutions to the more general case. The developed method of constructing a solution may be interesting for other areas of dynamo theory (e.g., the galactic dynamo and the geodynamo), which also consider the influence of advective flows that cannot be reduced to differential rotation.

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