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The Construction of Exact Solutions of the Model Problem on Rotating Fluid in Domains with Angular Points

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Abstract—A new method has been developed for the construction of a two-dimensional model problem on the oscillations of rotating ideal fluid in some domains containing angular points. It is proved that these solutions correspond to the absolutely continuous component of the spectrum of the linear operator connected with the problem.

Keywords: rotating fluid, no almost periodic solutions, absolutely continuous spectrum.

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INTRODUCTION

Investigations of the behavior of rotating fluids are increasingly important because the principle of fluid medium rotation is used for the solution of numerous problems in modern science and technology. Thus, the theory of rotating fluids is widely used in geophysical hydrodynamics to study the generation and propagation of vortex structures and flows in the ocean. In the chemical, food-processing, mining, and other industries the rotation of fluids is used to implement the processes of separation and to explore heterogeneous mixtures by the centrifuge method. Recently, special interest has been aroused by collapsars (black holes), whose behavior can be modeled by the behavior of a rotating fluid with an ultra-low viscosity. Many more examples can be added but at the same time, many theoretical issues connected with the behavior of a rotating ideal incompressible fluid are very far from clear.

The motion of an ideal fluid fully filling some uniformly rotating closed vessel Q is described by the following system of equations:

$$\frac{\partial \mathbf{U}}{\partial t} + 2\mathbf{k} \times \mathbf{U} = -\nabla p, \qquad \nabla \cdot \mathbf{U} = 0, \quad \mathbf{U} \cdot \mathbf{n}\big|_{\partial Q} = 0 \tag{1}$$

where $\mathbf{k}$ is the vector of the rotation axis, $\mathbf{U}$ is the deviation of the fluid velocity field from the uniform rotation, and p is the dynamic pressure. The investigation into the qualitative properties of the solutions of this system is known to be a difficult problem (at present it has been completely solved only for vessels shaped as straight circular cylinders and ellipsoids of rotation symmetrical about the rotation axis) and, therefore, the following model 2D problem is considered along with this system:

$$\frac{\partial u}{\partial t} = v - \frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -u, \quad \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}, \quad (x, z) \in D, \quad D \subset \mathbf{R}^2, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (x, z) \supseteq D, \tag{3}$$

$$u\big|_{t=0} = u_0, \quad v\big|_{t=0} = v_0, \quad w\big|_{t=0} = w_0, \quad (x, z) \in D, \tag{4}$$

$$un_1 + wn_3\big|_{\partial D} = 0, \tag{5}$$

in which it is assumed that the components of the velocity $\mathbf{U} = (u, v, w)$ and the pressure p only depend on the time and two space variables x and z; the domain occupied by the fluid is a cylinder $Q = \{(x, y, z)|(x, z) \in D, y \in \mathbf{R}\}$, $\mathbf{k} = (0, 0, 1/2)$ (which puts no constraint on the generality), and $\mathbf{n} = (n_1, n_3)$ is the vector of the normal to ∂D in the plane Oxz. At the same time, in order to eliminate the arbitrariness in the definition of the function of the pressure p it is usually assumed that p under every t satisfies the equality

$$\iint_D p\,dx\,dz = 0. \tag{6}$$

One of the main questions that arises in studying the problem (2)–(5) is the problem of the behavior of its solutions at $t \longrightarrow \infty$, which is known to be strongly dependent on the type of the domain D.

It can readily be shown that the flow function ψ [psi] corresponding to the system (2)–(5) determined by the equations

$$\frac{\partial\psi}{\partial x} = w, \quad \frac{\partial\psi}{\partial z} = -u, \tag{7}$$

is the solution of the following problem (see, for instance, [1])

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{\partial^2\psi}{\partial z^2} = 0, \tag{8}$$

$$(x, z; t) \in D \times (0, \infty),$$

$$\psi|_{\partial D \times (0, \infty)} = 0, \tag{9}$$

$$\psi|_{t=0} = \psi_0, \quad \psi_t|_{t=0} = \psi_1, \tag{10}$$

$$(\psi_0|_{\partial D} = 0, \psi_1|_{\partial D} = 0).$$

We denote by $\overset{\circ}{W}{}_2^1(D)$ the space that is the completion of the set $C_0^\infty(D)$ of infinitely differentiable functions with the supports contained in D in the norm generated by the inner product

$$(f, g)_1 = \iint_D (\nabla f, \nabla \bar{g}) dx dz. \tag{11}$$

In studying the problem (8)–(10) it would be necessary to introduce into consideration the following operator A acting in the space $\overset{\circ}{W}{}_2^1(D)$ because on the smooth functions $h \in C_0^\infty(D)$ it is determined as the solution of the problem

$$\Delta(Ah) = \frac{\partial^2 h}{\partial z^2}, \quad Ah \in \overset{\circ}{W}{}_2^1(D) \tag{12}$$

and then it expands at infinity to the limited operator in $\overset{\circ}{W}{}_2^1(D)$ self-conjugate with respect to the inner (scalar) product (11). By means of the operator A the problem (8)–(10) can be rewritten in the form of the abstract Cauchy problem

$$\Psi'' = -A\Psi, \quad \Psi(0) = \Psi_0, \quad \Psi'(0) = \Psi_1, \tag{13}$$

where $t \longrightarrow \Psi(x, y, t)$ is a function with the values in $\overset{\circ}{W}{}_2^1(D)$ and $\Psi_0, \Psi_1 \in \overset{\circ}{W}{}_2^1(D)$ (see, for instance, [2]). Therefore, it appears that the initial problem of the investigation into the behavior of a rotating fluid in the domain D is closely connected with studying the spectrum of the operator A.

It is known that for every domain D the spectrum of the operator A is the entire interval [0, 1] but its qualitative structure significantly depends on the type of the domain. The properties of the solutions of problem (13) depending on the domain configurations have been studied by many authors; for an extensive bibliographical list see [3] and [6]. However, even in this model two-dimensional case, this problem has been fully investigated only for rectangles and ellipses symmetrical with respect to Oz because for these domains the spectrum of the operator A is purely point and all the solutions of problem (2)–(5) are almost periodic functions of time. At the same time, it is rather difficult to prove the existence of domains for which the solutions of this problem are not almost periodic and it is even harder to construct the specific examples of such domains and to study the respective solutions because this involves exploration of the Dirichlet problem for the hyperbolic equation. In this work, a new method is proposed for the construction of the exact solutions of the problem (8)–(10): it is considered in detail for a certain space of initial vectors in the case where D is a triangle, i.e.,

$$D := \left\{(x, z) \middle| 0 < x < \frac{1}{\alpha}, 0 < z < \alpha x\right\}, \tag{14}$$

$$(0 < \alpha < +\infty).$$

This enables us to obtain the representations of some class of solutions for the problem (2)–(5) in explicit form. It is proved that these solutions correspond to the absolutely continuous component of the spectrum of the operator A. At the same time, as becomes clear from these considerations, this method is readily extended to a sufficiently wide class of the domains with angular points (see the note in the end of this publication).

1. THE SPECTRUM OF THE OPERATOR A AND DIFFERENTIAL SOLUTIONS

It can easily be seen that $\lambda \in (0, 1)$ then and only then is the eigenvalue of the operator A when the hyperbolic equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial^2\psi}{\partial z^2} = 0, \quad a^2 = \frac{\lambda}{1-\lambda} \tag{15}$$

has at least one nontrivial generalized solution $\psi \in \overset{\circ}{W}{}_2^1(D)$. From the author's works [7, 8] it follows that there are no such solutions for the considered domain D, and the spectrum of the operator A is absolutely continuous, i.e., $\sigma(A) = \sigma_c(A)$. Therefore, if E_λ is the spectral function of the operator A and h is an arbitrary element of the space $\overset{\circ}{W}{}_2^1(D)$, the function $(E_\lambda h, h)_1$ is continuous in [0, 1].

We consider the hyperbolic equation (15) in the domain D. Its characteristics of the form $x = az + c'$ will be referred to as the characteristics of the first family and the characteristics of the type $x = -az + c''$ will be called the characteristics of the second family. We

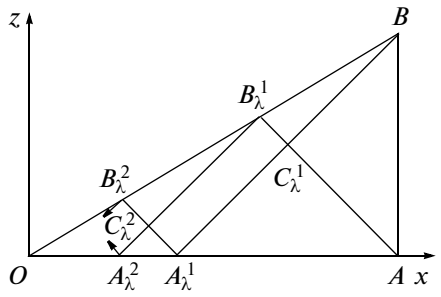

Fig. 1. The partition of the domain D.

fix the arbitrary value of $\lambda \in (0, (1+\alpha^2)^{-1})$. The following infinite polygonal curves will be defined by specifying the law regulating the reflection of the beams of characteristic directions from the boundary. We assume that beam emitted from the point $B(1/\alpha, 1)$ in the direction of the first family of the characteristics inward the domain D on meeting the boundary ∂D is reflected from it and then follows on inside D directed along the characteristics of the second family, then, on reaching the boundary ∂D, it is reflected from it and goes along the characteristics of the first family, and so on. It will also be assumed that the beam going out of $A(1,0)$ and directed along the first family of the characteristics inside the domain behaves similarly when reflected from ∂D. Note that under the considered values of $\lambda \in (0, (1+\alpha^2)^{-1})$ both beams therewith are localized in the angle with the vertex O. The polygonal curves, which are the trajectories of these beams, partition D into an infinite number of triangles T_λ^i and parallelograms P_λ^i. The vertices of these polygonal curves located in the interval $[OB]$ will be designated by B_λ^i and the vertices located in the interval $[OA],|-$ through A_λ^i, and the interior points of their intersections will be denoted by C_λ^i, $i = 1, 2, \ldots$, and numbered in succession in the direction from the interval $[AB]$ to the point $O(0, 0)$ (see Fig. 1).

Let $\theta_1 \in L_2(0, 1)$ be an arbitrary function. By this function, in the triangle ABC_λ^1 such a generalized solution $\psi_0(x, z, \lambda) \in W_2^1(ABC_\lambda^1)$ of Eq. (15) will be constructed that

$$\psi_0\big|_{AB} = 0, \quad \left.\frac{\partial\psi_0}{\partial x}\right|_{AB} = \theta_1.$$

It follows from the publication [11] that such a solution ψ_0 exists and is unique. Further, using the function ψ_0 in the triangle $BC_\lambda^1B_\lambda^1$ we construct such a generalized solution $\psi_1(x, z, \lambda) \in W_2^1(BC_\lambda^1B_\lambda^1)$ of Eq. (15) that

$$\psi_1\big|_{BC_\lambda^1} = \psi_0\big|_{BC_\lambda^1}, \quad \psi_1\big|_{BB_\lambda^1} = 0.$$

The existence and uniqueness of ψ_1 follows from [8]. Then, using the functions ψ_0 and ψ_1, in the triangle $AB_\lambda^1A_\lambda^2$ we construct such a generalized solution $\psi_2(x, z, \lambda) \in W_2^1(AB_\lambda^1A_\lambda^2)$ of Eq. (15) that

$$\psi_2\big|_{AC_\lambda^1} = \psi_0\big|_{AC_\lambda^1},$$

$$\psi_2\big|_{C_\lambda^1B_\lambda^1} = \psi_1\big|_{C_\lambda^1B_\lambda^1}, \quad \psi_2\big|_{AA_\lambda^2} = 0,$$

and so on. Let

$$\psi(x, z; \theta_1; \lambda) = \begin{cases} \psi_0(x, z, \lambda) \ \text{ at } \ (x, z) \in \Delta ABC_\lambda^1, \\ \psi_1(x, z, \lambda) \ \text{ at } \ (x, z) \in \Delta BC_\lambda^1B_\lambda^1, \\ \psi_2(x, z, \lambda) \ \text{ at } \ (x, z) \in \Delta AB_\lambda^1A_\lambda^1, \\ \ldots \qquad\qquad \ldots \end{cases} \tag{16}$$

It can readily be shown that for any $\theta_1 \in L_2(0, 1)$ and for all $\lambda \in (0, (1+\alpha^2)^{-1})$ the thus constructed function $\psi(x, z; \theta_1; \lambda)$ has the following properties:

a) The function $\psi(x, z; \theta_1; \lambda)$ belongs to $L_2(D)$ and is the generalized solution of Eq. (15);

b) For any $0 < \varepsilon < 1$ the function $\psi(x, z; \theta_1; \lambda)$ belongs to the space $W_2^1(D \cap \{x > \varepsilon\})$;

c) For any smooth curve $\mathbf{r}(t) := \{x(t), z(t), \lambda(t)\}$, $t \in [t_0, t_1]$ lying in the domain $D \times (0, (1+\alpha^2)^{-1})$ the function $\psi(x(t), z(t); \theta_1, \lambda(t))$ is absolutely continuous in $[t_0, t_1]$; and

d) For almost every $x \in (0, 1/\alpha)$ $\psi_x'(x, \alpha x; \theta_1; \lambda)$, $\psi_z'(x, \alpha x; \theta_1; \lambda)$, $\psi_x'(x, 0; \theta_1; \lambda)$, and $\psi_z'(x, 0; \theta_1; \lambda)$ are determined and for any $0 < \varepsilon < 1/\alpha$ there exists a constant C_ε independent of θ_1, such that

$$\|\psi_x'(x, \alpha x; \theta_1; \lambda)\|_{L_2(\varepsilon, 1/\alpha)} \le C_\varepsilon \|\theta_1\|_{L_2},$$

$$\|\psi_z'(x, \alpha x; \theta_1; \lambda)\|_{L_2(\varepsilon, 1/\alpha)} \le C_\varepsilon \|\theta_1\|_{L_2},$$

$$\|\psi_x'(x, 0; \theta_1; \lambda)\|_{L_2(\varepsilon, 1/\alpha)} \le C_\varepsilon \|\theta_1\|_{L_2},$$

$$\|\psi_z'(x, 0; \theta_1; \lambda)\|_{L_2(\varepsilon, 1/\alpha)} \le C_\varepsilon \|\theta_1\|_{L_2}.$$

The thus constructed function $\psi(x, z; \theta_1; \lambda)$, as can easily be seen, is a certain nontrivial solution of the Dirichlet problem for the hyperbolic equation (15) but it does not belong to the space $\overset{\circ}{W}{}_2^1(D)$ (it has a singularity at the point $O(0, 0)$), and, therefore, is not an eigenfunction of the operator A and does not generate the solution of the studied initial boundary value problem (2)–(5). It appears at the same time that being integrated over the spectral parameter λ, the obtained function will already belong to the space $\overset{\circ}{W}{}_2^1(D)$.

Specifically, let $0 < \lambda_* < \lambda_{**} < (1 + \alpha^2)^{-1}$ and let the function $\sigma(\mu) \in C^1[\lambda_*, \lambda_{**}]$. Then for any λ_1, λ_2, such that $\lambda_* \le \lambda_1 < \lambda_2 \le \lambda_{**}$, the function

$$\Upsilon(x, z; \theta_1; \sigma) := \int_{\lambda_1}^{\lambda_2} \sigma(\mu)\psi(x, z; \theta_1; \mu)d\mu \qquad (17)$$

belongs to the space $\overset{\circ}{W}{}_2^1(D)$ and the following estimate holds true:

$$\|\Upsilon(x, z; \theta_1; \sigma)\|_1 \le M(\lambda_2 - \lambda_1)\|\theta_1\|_{L_2}, \qquad (18)$$

where M does not depend on λ_1, λ_2, and θ_1.

The last fact is surprising and it is this point that is the key point of the present work, because this new function will further make it possible to construct the exact solutions of the problem (2)–(5). We now turn to the main points of its proof.

In order to prove that $\Upsilon(x, z; \theta_1; \sigma) \in \overset{\circ}{W}{}_2^1(D)$ by virtue of the property b) it is sufficient to show that

$$\|\Upsilon'_x\|_{L_2(\mathcal{D}_2)} < \infty, \quad \|\Upsilon'_z\|_{L_2(\mathcal{D}_2)} < \infty,$$

where $\mathcal{D}_2 := D \cap \{a_2 z > x + a_2 - 1/\alpha\}$. $a_2 = \sqrt{\dfrac{\lambda_2}{1 - \lambda_2}}$. For $(x, z) \in \mathcal{D}_2$ for all $\mu \in [\lambda_1, \lambda_2]$ from the Riemann formula we have (see [7])

$$\psi(x, z; \theta_1; \mu) = \frac{\sqrt{\mu}}{2\sqrt{1 - \mu}} \int_{P(x, z; \mu)}^{Q(x, z; \mu)} \varphi(x', \mu)dx', \qquad (19)$$

where

$$\varphi(x, \mu) := \left(\alpha\psi'_x + \frac{1 - \mu}{\mu}\psi'_z\right)\Bigg|_{z = \alpha x}, \qquad (20)$$

and $P(x, z; \mu)$ and $Q(x, z; \mu)$ are the abscissas of the right and left angles of the characteristic triangle with the vertex at the point (x, z) corresponding to the value $\lambda = \mu$ and resting on OB. It can readily be found that

$$P(x, z; \mu) = \frac{\alpha x - z}{2\alpha} l + \frac{\alpha x + z}{2\alpha},$$
$$Q(x, z; \mu) = \frac{\alpha x + z}{2\alpha} + \frac{\alpha x - z}{2\alpha l}, \qquad (21)$$

where

$$l = l(\mu) := \frac{\sqrt{1 - \mu} + \alpha\sqrt{\mu}}{\sqrt{1 - \mu} - \alpha\sqrt{\mu}} \qquad (22)$$

is a function strictly monotonously increasing in the interval $(0, (1 + \alpha^2)^{-1})$ assuming the values from the interval $(1, +\infty)$, therefore,

$$l(\mu) > 1 \quad \text{for every} \quad \mu \in [\lambda_1, \lambda_2]. \qquad (23)$$

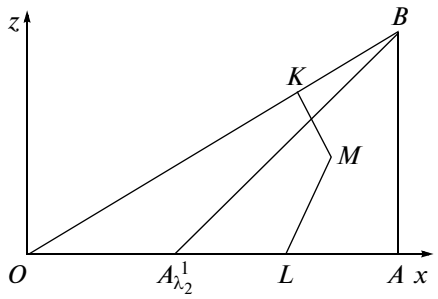

Fig. 2. Polygon $T(x, z, \lambda)$.

Thus, for $(x, z) \in \mathcal{D}_2$ we have

$$\Upsilon(x, z; \theta_1; \sigma) = \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\mu}}{2\sqrt{1 - \mu}} \left\{ \int_{P(x, z; \mu)}^{Q(x, z; \mu)} \varphi(\tilde{x}, \mu)d\tilde{x} \right\} d\mu,$$

therefore,

$$\|\Upsilon'_x\|^2_{L_2(\mathcal{D}_2)}$$

$$= \iint_{\mathcal{D}_2} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\mu}}{2\sqrt{1 - \mu}} \{Q'_x\varphi(Q, \mu) - P'_x\varphi(P, \mu)\} d\mu \right|^2 dxdz \qquad (24)$$

$$\le 2 \iint_{\mathcal{D}_2} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\mu}}{2\sqrt{1 - \mu}} P'_x\varphi(P, \mu)d\mu \right|^2 dxdz$$

$$+ 2 \iint_{\mathcal{D}_2} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\mu}}{2\sqrt{1 - \mu}} Q'_x\varphi(Q, \mu)d\mu \right|^2 dxdz.$$

The proof of the existence of these integrals for any function $\theta_1 \in L_2(0, 1)$ and any $\sigma(\mu) \in C^1[\lambda_*, \lambda_{**}]$ is based on the fact that the set of all the piecewise constant functions specified in the interval (0, 1), such that the end points of the intervals of constancy an interval constancy are rational numbers, is dense (tight) everywhere in $L_2(0, 1)$, therefore, if as $\theta_1(x)$ the function

$$\theta_{1, n}(x) = \begin{cases} \dots & \dots \\ c_i, & \dfrac{i - 1}{n} < x < \dfrac{i}{n}, \\ \dots & \dots \end{cases} \qquad (25)$$

is taken where $c_i \in \mathbf{C}$ are some numbers, $i = 1, 2, \dots, n$, then the function φ can be presented in the explicit form. This makes it possible to find that

$$\iint\limits_{\mathcal{D}_2}\left|\int\limits_{\lambda_1}^{\lambda_2}\sigma(\mu)\frac{\partial P}{\partial x}(x, z; \mu)\varphi(P(x, z; \mu), \mu)d\mu\right|^2 dxdz$$
$$\le C_1(l_2 - l_1)^2\|\theta_1(x)\|^2_{L_2},$$

where C_1 is some constant. Obviously, for the first integral in (24) a similar estimate will hold true, therefore,

$$\|V_x'\|^2_{L_2(\mathcal{D}_2)} \le C_2(\lambda_2 - \lambda_1)^2\|\theta_1(x)\|^2_{L_2}, \tag{26}$$

where C_2 does not depend on θ_1, λ_1, λ_2, hence it follows that the function $\Upsilon(x, z; \theta_1, \sigma)$ belongs to the space $\overset{\circ}{W}{}^1_2(D)$.

In order to prove the stated estimate for its norm in $\overset{\circ}{W}{}^1_2(D)$ we now evaluate the quantity $\|\Upsilon_x'\|^2_{L_2(D\setminus\overline{\mathcal{D}_2})}$. Let $M(x, z) \in \{D\setminus\overline{\mathcal{D}_2}\}$, i.e., $M(x, z) \in DAA_{\lambda_2}B$ (see Fig. 2). Through this point we construct the characteristics of Eq. (15) of the first family up to the intersection with the segment OA and the obtained intersection point will be designated by $L(x - az; 0)$. Then through the point $M(x, z)$ we construct the characteristics of the second family up to the intersection with the segment OB and denote the obtained intersection point by $K\left(\frac{x + az}{1 + a\alpha}; \alpha\frac{x + az}{1 + a\alpha}\right)$. Note that from the properties b), c), and d) of the function $\psi(x, z; \theta_1; \lambda)$ it follows that for the polygon $T(x, z, \lambda)$ with the vertices at the points M, K, B, A, L

$$\int\limits_{\partial T(x, z; \lambda)}\frac{\partial\psi}{\partial x}dz + \frac{1}{a^2}\frac{\partial\psi}{\partial z}dx$$
$$= \iint\limits_{T(x, z; \lambda)}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\psi}{\partial z^2}\right)dxdz = 0$$

is satisfied. Hence follows the equality

$$\psi|_M = \psi(x, z; \theta_1; \lambda) = -\frac{a}{2}\int\limits_{x - az}^{1/\alpha}\varphi_0(t, \lambda)dt$$
$$+\frac{a}{2}\int\limits_A^B\theta_1(t)dt + \frac{a}{2}\int\limits_{1/\alpha}^{\frac{x + az}{1 + a\alpha}}\varphi(t, \lambda)dt, \tag{27}$$

where $\varphi_0(t, \lambda) := \frac{1}{a^2}\frac{\partial\psi}{\partial z}\Big|_{z = 0}$. Therefore, for almost every $(x, z) \in \{D\setminus\overline{\mathcal{D}_2}\}$

$$\Upsilon_x' = \int\limits_{\lambda_1}^{\lambda_2}\psi(x, z; \theta_1; \lambda)_x'd\lambda$$
$$= \int\limits_{\lambda_1}^{\lambda_2}\frac{a}{2}\left(\varphi_0(x - az, \lambda) + \frac{1}{1 + a\alpha}\varphi\left(\left(\frac{x + az}{1 + a\alpha}\right), \lambda\right)\right)d\lambda. \tag{28}$$

Then

$$\|\Psi_x'\|^2_{L_2(D\setminus\overline{\mathcal{D}_2})} = \iint\limits_{D\setminus\overline{\mathcal{D}_2}}\left\{\int\limits_{\lambda_1}^{\lambda_2}\frac{a}{2}\left(\varphi_0(x - az, \lambda)\right.\right.$$
$$\left.\left. + \frac{1}{1 + a\alpha}\varphi\left(\frac{x + az}{1 + a\alpha}, \lambda\right)\right)d\lambda\right\}^2 dxdz$$
$$\le (\lambda_2 - \lambda_1)\iint\limits_{D\setminus\overline{\mathcal{D}_2}}\int\limits_{\lambda_1}^{\lambda_2}\frac{a^2}{4}\left|\varphi_0(x - az, \lambda)\right.$$
$$\left. + \frac{1}{1 + a\alpha}\varphi\left(\frac{x + az}{1 + a\alpha}, \lambda\right)\right|^2 d\lambda dxdz$$
$$\le M_0(\lambda_0 - \lambda_1)\int\limits_{\lambda_1}^{\lambda_2}\left\{\iint\limits_{D\setminus\overline{\mathcal{D}_2}}\left|\varphi_0(x - az, \lambda)\right|^2\right.$$
$$\left. + \left|\varphi\left(\frac{x + az}{1 + a\alpha}, \lambda\right)\right|^2 dxdz\right\}d\lambda \tag{29}$$
$$\le M_1(\lambda_2 - \lambda_1)\int\limits_{\lambda_1}^{\lambda_2}\left\{\|\varphi_0\|^2_{L_2(A^1(\lambda_2)A)} + \|\varphi\|^2_{L_2(B^2(\lambda_2)B)}\right\}d\lambda$$
$$\le M_2(\lambda_2 - \lambda_1)^2\|\theta_1\|^2_{L_2},$$

where the positive constants M_i, $i = 0, 1, 2$ are independent of λ_1, λ_2, θ_1 Taking (26) into account we obtain

$$\|\Upsilon_x'\|^2_{L_2(D)} \le M_3(\lambda_2 - \lambda_1)^2\|\theta_1\|^2_{L_2},$$

where the constant M_3 does not depend on λ_1, λ_2, θ_1. Evidently, the estimate for $\|\Upsilon_z'\|^2_{L_2(D\setminus\overline{\mathcal{D}_2})}$ is similar. Thus, estimate (18) has been proved.

Let us see now how this can be used for the investigation into the structure of the spectrum of the operator A.

We define the function

$$\Psi(x, z; \theta_1; \sigma; \lambda)$$

$$:= \begin{cases} \mathbf{0}, & \text{at } \lambda \le \lambda_*, \\ \int\limits_{\lambda_*}^{\lambda} \sigma(\mu)\psi(x, z; \theta_1; \mu)d\mu, & \text{at } \lambda_* < \lambda \le \lambda_{**} \\ \int\limits_{\lambda_*}^{\lambda_{**}} \sigma(\mu)\psi(x, z; \theta_1; \mu)d\mu, & \text{at } \lambda_{**} < \lambda, \end{cases} \quad (30)$$

where $\mathbf{0}$ is the zeroth element of the space $\overset{\circ}{W}{}_2^1(D)$.

From the proof above there follows the continuity and even the absolute continuity of the function $\|\Psi(x, z; \theta_1; \sigma; \lambda)\|$ with respect to λ.

Now, we prove that for every $\lambda_1, \lambda_2 \in (0, (1+\alpha^2)^{-1})$ the following equality will hold true:

$$A(\Psi(\lambda_2) - \Psi(\lambda_1)) = \int\limits_{\lambda_1}^{\lambda_2} \lambda d\Psi(\lambda). \quad (31)$$

Since $\Psi(x, z; \theta_1; \sigma; \lambda_2) - \Psi(x, y; \theta_1; \sigma; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda$ and $d\Psi(x, z; \theta_1; \sigma; \lambda) = \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda$, our task is to prove the equality

$$A\int\limits_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda = \int\limits_{\lambda_1}^{\lambda_2} \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda,$$

or, which is the same, after applying the Laplace operator to both its sides, we will have to prove the equality

$$\frac{\partial^2}{\partial z^2}\int\limits_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda = \Delta\int\limits_{\lambda_1}^{\lambda_2} \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda$$

in the space $W_2^{-1}(D)$. This implies that for an arbitrary $g \in C_0^\infty(D)$ it must be proved that

$$\iint\limits_D \left\{ \frac{\partial^2}{\partial z^2}\int\limits_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda g - \Delta\int\limits_{\lambda_1}^{\lambda_2} \lambda\sigma(\lambda)u(x, y; \theta_1; \lambda)d\lambda g \right\} dxdz = 0.$$

Since by the construction the function $\sigma(\lambda)\psi(x, z; \theta_1; \lambda) \in W_2^1(D \cap \{x > \varepsilon\})$, $\varepsilon > 0$ is the generalized solution of Eq. (15), for every $g \in C_0^\infty(D)$ we have

$$\iint\limits_D \sigma\psi\frac{\partial^2 g}{\partial z^2}dxdz = \lambda\iint\limits_D \sigma\psi\Delta g dxdz.$$

Therefore,

$$\iint\limits_D \left\{ \frac{\partial^2}{\partial z^2}\int\limits_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda g - \Delta\int\limits_{\lambda_1}^{\lambda_2} \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda g \right\} dxdz$$

$$= \iint\limits_D \left\{ \int\limits_{\lambda_1}^{\lambda_2} \sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda\frac{\partial^2 g}{\partial z^2} - \int\limits_{\lambda_1}^{\lambda_2} \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)d\lambda\Delta g \right\} dxdz$$

$$= \iint\limits_D \int\limits_{\lambda_1}^{\lambda_2} \left\{ \sigma(\lambda)\psi(x, z; \theta_1; \lambda)\frac{\partial^2 g}{\partial z^2} - \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)\Delta g \right\} d\lambda dxdz$$

$$= \int\limits_{\lambda_1}^{\lambda_2}\iint\limits_D \left\{ \sigma(\lambda)\psi(x, z; \theta_1; \lambda)\frac{\partial^2 g}{\partial z^2} - \lambda\sigma(\lambda)\psi(x, z; \theta_1; \lambda)\Delta g \right\} dxdz d\lambda = 0.$$

The above proof implies that there exists such a vector $h \in \overset{\circ}{W}{}_2^1(D)$ that $\Psi(\lambda) = E_\lambda h$ where E_λ is the spectrum function of the operator A, i.e., $\Psi(\lambda)$ is the differential solution of the spectral equation for the operator A (see [10]).

Further, let $\theta_2(x) \in L_2(0, 1/\alpha)$ be an arbitrary function. Similarly to the construction of the function $\psi(x, z; \theta_1; \lambda)$ for $\lambda \in (0, (1+\alpha^2)^{-1})$ we construct the function $\chi(x, z, \theta_2; \lambda)$ for $\lambda \in ((1+\alpha^2)^{-1}, 1)$: the function χ is the generalized solution of Eq. (15) belonging to $W_2^1(D \cap \{z < 1 - \varepsilon\})$ for every $0 < \varepsilon < 1$ and satisfying

the relations $\chi|_{\partial D} = 0$ and $\chi_z|_{[OA]} = \theta_2$. We consider the function

$$\tau(x, z; \lambda) := \begin{cases} \psi(x, z; \theta_1; \lambda), & \text{at } 0 < \lambda < (1+\alpha^2)^{-1} \\ \chi(x, z; \theta_2; \lambda), & \text{at } (1+\alpha^2)^{-1} < \lambda < 1. \end{cases} \quad (32)$$

It follows from the previous considerations that if $\sigma(\lambda)$ is an arbitrary function from $C^1(0, 1)$, such that its support $\operatorname{supp}\sigma$ is contained in the union of the intervals $(0, (1+\alpha^2)^{-1})$, then the function:

$$\Psi(x, z; \lambda) := \int_0^\lambda \sigma(\mu)\tau(x, z; \mu)d\mu, \quad \lambda \in [0, 1] \quad (33)$$

with the values in the space $\overset{\circ}{W}{}_2^1$ is the differential solution of the spectral equation for the operator A.

Moreover, if now we designate by H_0 the closure of the linear shell of all the differential solutions of the spectral equation for the operator A of the type (33), it follows from the above-presented proof that on H_0 the spectrum of the operator A is absolutely continuous, i.e., $\sigma(A|_{H_0}) = \sigma_{ac}(A|_{H_0})$.

2. THE SOLUTIONS OF THE PROBLEM (2)–(5)

By means of the above-constructed differential solutions $\Psi(x, z; \lambda)$ of the type (33) it is now possible to write out the exact solutions of the problems (8)–(10) and (2)–(5). Namely, let

$$\tilde{\psi}_0(x, z) := \int_0^1 \sigma_0(\lambda)\tau_0(x, z; \lambda)d\lambda,$$
$$\tilde{\psi}_1(x, z) := \int_0^1 \sigma_1(\lambda)\tau_1(x, z; \lambda)d\lambda, \quad (34)$$

where τ_0 and τ_1 are constructed by the above-described method on the basis of the functions $\theta_1^{(0)}$, $\theta_2^{(0)}$, $\theta_1^{(1)}$, and $\theta_2^{(1)}$, respectively, a and the supports $\operatorname{supp}\sigma_i$ are contained in the union of the intervals $(0, (1+\alpha^2)^{-1})$ and $((1+\alpha^2)^{-1}, 1)$. Then the function

$$\tilde{\psi}(x, z; t) := \int_0^1 \cos(\sqrt{\lambda}t)\sigma_0(\lambda)\tau_0(x, z; \lambda)d\lambda + \int_0^1 \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\sigma_1(\lambda)\tau_1(x, z; \lambda)d\lambda \quad (35)$$

is the solution of the problem (8)–(10), and at the same time $\|\tilde{\Psi}(x, z; t)\|_{L_2(D)} \longrightarrow 0$ at $t \longrightarrow \infty$.

An explanation here is only required for the last statement because as in H_0 the spectrum of the operator A is absolutely continuous, for any element g of the space $\overset{\circ}{W}{}_2^1(D)$ the function $(\tilde{\psi}, g)_1$ tends to zero at $t \longrightarrow \infty$, hence there follows the decrease in the L_2-norm of this solution. Finally, we can present the class of exact solutions of the initial model problem (2)–(5) describing the oscillations of a rotating fluid. Let

$$u_0(x, z) = -\int_0^1 \sigma_0(\lambda)\frac{\partial\tau_0}{\partial z}(x, z; \lambda)d\lambda,$$
$$w_0(x, z) = \int_0^1 \sigma_0(\lambda)\frac{\partial\tau_0}{\partial x}(x, z; \lambda)d\lambda,$$

$v_0(x, z)$ is the solution of the equation

$$\frac{\partial v_0}{\partial z} = -\Delta\int_0^1 \sigma_1(\lambda)\tau_1(x, z; \lambda)d\lambda, \quad (36)$$

then the vector function $U := (u, v, w)$ where

$$u = -\frac{\partial\tilde{\psi}}{\partial z},$$
$$v = \int_0^t \frac{\partial\tilde{\psi}}{\partial z}(x, z; s)ds + v_0(x, z), \quad w = \frac{\partial\tilde{\psi}}{\partial x}, \quad (37)$$

is the solution of the problem (2)–(5).

It can readily be seen that the function $v_0(x, z)$ is found from (36) with an accuracy to the term $v_*(x)$. At the same time, the function of the hydrodynamic pressure $p(x, z, t)$ with an accuracy to the term $p^*(x) = \int v_*(x)dx$ can be obtained from the system

$$\frac{\partial p}{\partial x} = \int_0^t \frac{\partial\tilde{\psi}}{\partial z}(x, z; s)ds + \frac{\partial}{\partial t}\frac{\partial\tilde{\psi}}{\partial z} + v_0(x, z),$$
$$\frac{\partial p}{\partial z} = -\frac{\partial}{\partial t}\frac{\partial\tilde{\psi}}{\partial x}. \quad (38)$$

Therefore, with an accuracy to the stationary solutions of the problem we have obtained the explicit representations for the velocity components of the rotating fluid in the studied model case.

In conclusion, it should be noted that the proposed method for constructing the exact solutions of the problem (2)–(5) can be used for a certain sufficiently wide class of domains with angular points. Let, for example, D be a "curvilinear triangle," the sides OA and OB of which are some smooth curves intersecting at the point O at a nonzero angle. If for all $\lambda \in (\lambda', \lambda'')$ the beams of the characteristic directions, whose law of reflection from the boundaries is described in Section 1, are localized in the angle with the vertex O, then

in this interval (λ', λ'') the differential solutions can be constructed of the spectral equation for the operator A similarly to the way it was done in Section 1 and the solutions of the nonstationary problem (8)–(10) corresponding to them.

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