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Solution Properties of a Model Problem on Oscillations of a Rotating Fluid in Domains with Angular Points

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Abstract—The behavior of a two-dimensional model problem on the oscillations of a rotating ideal fluid in some domains containing angular points has been studied under $t \longrightarrow \infty$. The existence is proved of solutions for which the entire energy of the initial state with time is almost completely concentrated in arbitrarily small neighborhoods of the angular points.

Keywords: rotating fluid, no almost periodic solutions, energy distribution.

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INTRODUCTION

The study of rotating fluid behavior is an important problem, whose significance is currently increasing for several reasons. In the first place, the "classic" fields of its application, such as geophysical hydrodynamics, are witnessing considerably enhanced and rapidly developing means and systems for data collection and processing, which has brought the problems on the investigation into the generation and propagation of vortex currents and flows much closer to the operational forecasting problems. Secondly, the development ofphysical science led to the intensive theoretical and experimental studies in such fields as astrophysics, high-energy physics, the general theory of relativity, and quantum cosmology, where the model of a rotating fluid is one of the basic issues. And, finally, problems considered by technology and connected, for instance, with the centrifugal process, have not lost their significance. At the same time, many theoretical questions related to the behavior of a rotating ideal incompressible fluid are far from clear.

The present study is devoted to the behavior at $t \longrightarrow \infty$ of a certain class of solutions of the following 2D-model problem on the oscillations of an ideal fluid rotating around the axis Oz:

$$\frac{\partial u}{\partial t} = v - \frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -u,$$
$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}, \quad (x, z) \in D, \quad D \subset \mathbf{R}^2, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (x, z) \in D, \tag{2}$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0,$$
$$w|_{t=0} = w_0, \quad (x, z) \in D, \tag{3}$$

$$un_1 + wn_3|_{\partial D} = 0. \tag{4}$$

It is assumed that the domain occupied by the fluid is a cylinder $Q = \{(x, y, z) | (x, z) \in D, y \in \mathbf{R}\}$ where the domain D is a triangle

$$D := \left\{ (x, z) \middle| 0 < x < \frac{1}{\alpha}, 0 < z < \alpha x \right\}, \tag{5}$$
$$(0 < \alpha < +\infty).$$

The deflection of the fluid velocity field from the uniform rotation $\mathbf{U} = (u, v, w)$ and the dynamic pressure p only depend on the time t and two space variables x and z, $\mathbf{n} = (n_1, n_3)$ is the vector of the normal to ∂D in the plane Oxz, and the function p at all t satisfies the equality $\iint_D p\, dxdz = 0$.

The studied class of solutions in the explicit form was constructed in [1] by means of the integral representations of the flow function ψ corresponding to this system. In the present study, it is shown that under a certain choice of the initial perturbation vector, the deflections of the fluid velocity components u, v, w from the uniform rotation at $t \longrightarrow \infty$ tend to zero at each interior point of the domain occupied by the fluid, while the energy proportion of the part of the fluid outside arbitrarily small neighborhoods of the edges of the cylinder Q that correspond to the vertices of acute angles in the triangle D tends to zero when t increases.

The denotations introduced in [1] will be used below.

1. THE PROPERTIES OF THE SMOOTH SOLUTIONS OF PROBLEM (1)–(4)

In the domain D we consider the hyperbolic equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\psi}{\partial z^2} = 0, \quad a^2 = \frac{\lambda}{1-\lambda}. \tag{6}$$

Its characteristics of the type $x = az + c'$ will be called the characteristics of the first family and the characteristics of the type $x = -az + c''$ will be referred to as the characteristics of the second family. We fix the arbitrary value $\lambda \in (0, (1+\alpha^2)^{-1})$. The following infinite polygonal curves will be defined by specifying the law regulating the reflection of the beams of characteristic directions from the boundary. We assume that the beam emitted from the point $B(1/\alpha, 1)$ in the direction of the first family of the characteristics inward of domain D on meeting the boundary ∂D is reflected from it and then follows on inside D, being directed along the characteristics of the second family; then, on reaching the boundary ∂D, the beam is reflected from it and goes along the characteristics of the first family, and so on.

It will also be assumed that the beam exiting $A(1, 0)$ and directed along the first family of the characteristics inside the domain behaves similarly when reflected from ∂D. Note that under the considered values of $A(1, 0)$ both beams are localized in the angle with the vertex O. The polygonal curves, which are the trajectories of these beams, partition D into an infinite number of triangles T_λ^i and parallelograms P_λ^i. The vertices of these polygonal curves located in the interval $[OB]$ will be designated by B_λ^i and the vertices located in the interval $[OA]$ through A_λ^i, and the interior points of their intersections will be denoted by C_λ^i, $i = 1, 2, \ldots$, and numbered in succession in the direction from the interval $[AB]$ to the point $O(0, 0)$.

Let $\theta_1 \in L_2(0, 1)$ be an arbitrary function. By this function, in the triangle ABC_λ^1 such a generalized solution $\psi_0(x, z, \lambda) \in W_2^1(ABC_\lambda^1)$ of Eq. (6) will be constructed that

$$\psi_0\big|_{AB} = 0, \quad \left.\frac{\partial\psi_0}{\partial x}\right|_{AB} = \theta_1.$$

It follows from [6] that such a solution ψ_0 exists and is unique. Further, using the function ψ_0 in the triangle $BC_\lambda^1B_\lambda^1$ we construct such a generalized solution $\psi_1(x, z, \lambda) \in W_2^1(BC_\lambda^1B_\lambda^1)$ of Eq. (6) that $\psi_1\big|_{BC_\lambda^1} = \psi_0\big|_{BC_\lambda^1}$, $\psi_1\big|_{BB_\lambda^1} = 0$. The existence and uniqueness of ψ_1 follows from [6]. Then, using the functions ψ_0 and ψ_1, in the triangle $AB_\lambda^1A_\lambda^2$ we construct such a generalized solution $\psi_2(x, z, \lambda) \in W_2^1(AB_\lambda^1A_\lambda^2)$ of Eq. (6), that

$$\psi_2\big|_{AC_\lambda^1} = \psi_0\big|_{AC_\lambda^1}, \quad \psi_2\big|_{C_\lambda^1B_\lambda^1} = \psi_1\big|_{C_\lambda^1B_\lambda^1}, \quad \psi_2\big|_{AA_\lambda^2} = 0.$$

and so on. Let

$$\psi(x, z; \theta_1; \lambda) := \begin{cases} \psi_0(x, z, \lambda) & \text{at } (x, z) \in \Delta ABC_\lambda^1, \\ \psi_1(x, z, \lambda) & \text{at } (x, z) \in \Delta BC_\lambda^1B_\lambda^1, \\ \psi_2(x, z, \lambda) & \text{at } (x, z) \in \Delta AB_\lambda^1A_\lambda^2, \\ \ldots & \ldots \end{cases} \tag{7}$$

Let $\theta_2(x) \in L_2(0, 1/\alpha)$ be an arbitrary function. Similarly to the construction of the function $\psi(x, z; \theta_1; \lambda)$ for $\lambda \in (0, (1+\alpha^2)^{-1})$, we construct the function $\chi(x, z; \theta_2; \lambda)$ for $\lambda \in ((1+\alpha^2)^{-1}, 1)$: the function χ is the generalized solution of Eq. (6) belonging to $W_2^1(D \cap \{z < 1 - \varepsilon\})$ for every $0 < \varepsilon < 1$ and satisfying the relations $\chi\big|_{\partial D} = 0$ and $\chi_z\big|_{[OA]} = \theta_2$. We consider the function

$$\tau(x, z; \lambda) = \begin{cases} \psi(x, z; \theta_1; \lambda), & \text{at } 0 < \lambda < (1+\alpha^2)^{-1}, \\ \chi(x, z; \theta_2; \lambda), & \text{at } (1+\alpha^2)^{-1} < \lambda < 1. \end{cases} \tag{8}$$

Let

$$u_0(x, z) = -\int_0^1 \sigma(\lambda)\frac{\partial\tau}{\partial z}(x, z; \lambda)d\lambda,$$

$$v_0(x, z) = 0, \quad w_0(x, z) = \int_0^1 \sigma(\lambda)\frac{\partial\tau}{\partial x}(x, z; \lambda)d\lambda,$$

where the support $\operatorname{supp}\sigma$ is contained in the union of intervals $(0, (1+\alpha^2)^{-1})$ and $((1+\alpha^2)^{-1}, 1)$ while, as has been proved in [1], the vector function $\mathbf{U} := (u, v, w)$ with the components

$$u = -\frac{\partial\tilde{\psi}}{\partial z}, \quad v = \int_0^t \frac{\partial\tilde{\psi}}{\partial z}(x, z; s)ds, \quad w = \frac{\partial\tilde{\psi}}{\partial x}, \tag{9}$$

where

$$\tilde{\psi}(x, z; t) := \int_0^1 \cos(\sqrt{\lambda}t)\sigma(\lambda)\tau(x, z; \lambda)d\lambda, \tag{10}$$

is the solution of the problem (1)–(4).

Let $\theta_1 \in C_0^\infty[0, 1]$, $\theta_2 \in C_0^\infty[0, 1/\alpha]$, $\sigma(\lambda) \in C_0^\infty[0,1]$, the support $\operatorname{supp}\sigma$ be contained in the union of the intervals $(0, (1+\alpha^2)^{-1})$ and $((1+\alpha^2)^{-1}, 1)$. Then for every $n = 1, 2, \ldots$ there exists such a constant $\mathcal{H}_n$

independent of t that under every $t \in (0, \infty)$ the following estimate holds true:

$$\|\tilde{\psi}(x, z; t)\|_{L_2(D)} \le \frac{\mathcal{H}_n}{t^n}, \quad (11)$$

where $\tilde{\psi}(x, z; t)$ is found in [11].

In our investigation, this fact is of major significance. Obviously, to prove this it is sufficient to consider the case where $\theta_2 \equiv 0$, $\operatorname{supp}\sigma \subset [\lambda_*, \lambda_{**}] \subset (0, (1+\alpha^2)^{-1})$, i.e.,

$$\tilde{\psi}(x, y; t) = \int_0^1 \cos(\sqrt{\lambda}t)\sigma(\lambda)\tau(x, z; \lambda)d\lambda = \int_0^{(1+\alpha^2)^{-1}} \cos(\sqrt{\lambda}t)\sigma(\lambda)\tau(x, z; \lambda)d\lambda, \quad (12)$$

where $\tau(x, z; \lambda) = \psi(x, z; \theta_1; \lambda)$. The function $\tau(x, z; \lambda)$ is indefinitely differentiable with respect to x, z, and λ is universally inside the prism $D \times (0, (1+\alpha^2)^{-1})$ and can be continuously extended with all of its derivatives over the entire surface of this prism, except the straight line $x = 0$, $z = 0$.

Let $\mathcal{D}_{**} := D \cap \{a_{**}z > x + a_{**} - 1/\alpha\}$, $a_{**} = \sqrt{\dfrac{\lambda_{**}}{1-\lambda_{**}}}$ for all $(x, z) \in \mathcal{D}_{**}$ and from Riemann's formula we have (see [2]):

$$\psi(x, z; \theta_1; \mu) = \frac{\sqrt{\mu}}{2\sqrt{1-\mu}} \int_{P(x, z; \mu)}^{Q(x, z; \mu)} \varphi(x', \mu)dx', \quad (13)$$

where

$$\varphi(x, \mu) := \left(\alpha\psi'_x + \frac{1-\mu}{\mu}\psi'_z\right)\Big|_{z=\alpha x}, \quad (14)$$

and $P(x, z; \mu)$ and $Q(x, z; \mu)$ are the abscissas of the right and left angles of the characteristic triangle with the vertex at the point (x, z) corresponding to the value $l = \mu$ and resting on OB

$$P(x, z; \mu) = \frac{\alpha x - z}{2\alpha}l + \frac{\alpha x + z}{2\alpha},$$
$$Q(x, z; \mu) = \frac{\alpha x + z}{2\alpha} + \frac{\alpha x - z}{2\alpha l}, \quad (15)$$

where

$$l = l(\mu) := \frac{\sqrt{1-\mu} + \alpha\sqrt{\mu}}{\sqrt{1-\mu} - \alpha\sqrt{\mu}}, \quad (16)$$

is a function that is strictly monotonously increasing in the interval $(0, (1+\alpha^2)^{-1})$ assuming the values from the interval $(1, +\infty)$, therefore,

$$l(\mu) > 1 \quad \text{for any} \quad \mu \in [\lambda_*, \lambda_{**}]. \quad (17)$$

It is clearly seen that in our conditions the function $\varphi(x, l)$ is indefinitely differentiable with respect to x, l everywhere inside the half-band $(0, 1/\alpha) \times (1, \infty)$. Let $l_1 := l(\lambda_*)$, $l_2 := l(\lambda_{**})$. Then, evidently, $1 < l_1 \le l_2 < +\infty$. It follows from the results of [1] that for every $n = 0, 1, 2, \ldots$, the following estimates will hold:

$$\left|\frac{\partial^n\tilde{\varphi}}{\partial l^n}\right| \le C(n)k^n l^k, \quad \frac{1}{\alpha l^{k+1}} < x \le \frac{1}{\alpha l^k}, \quad l_* \le l \le l_{**}, \quad (18)$$

where $k = 1, 2, \ldots$ and the constant $C(n)$ does not depend on k and (x, l).

By using these estimates we prove the estimate (11) at $n = 1$. We have

$$\tilde{\psi}(x, z; t) = \int_0^1 \cos(\sqrt{\lambda}t)\sigma(\lambda)\tau(x, z; \lambda)d\lambda = \int_0^1 \cos(\nu t)2\nu\sigma(\nu^2)\tau(x, z; \nu^2)d\nu$$
$$= -\frac{1}{t}\int_0^1 \sin(\nu t)(2\nu\sigma(\nu^2))'_\nu\tau(x, z; \nu^2)d\nu - \frac{1}{t}\int_0^1 \sin(\nu t)2\nu\sigma(\nu^2)\tau'_\nu(x, z; \nu^2)d\nu. \quad (19)$$

We prove the boundedness of L_2, i.e., the norm of the first integral

$$\iint_D \left\{\int_0^1 \sin(\nu t)(2\nu\sigma(\nu^2))'_\nu\tau(x, z; \nu^2)d\nu\right\}^2 dxdz \le \iiint_{D\,0}^{\quad 1} \left|(2\nu\sigma(\nu^2))'_\nu\tau(x, z; \nu^2)\right|^2 d\nu dxdz$$
$$= \int_0^1 \iiint_D \left|(2\nu\sigma(\nu^2))'_\nu\tau(x, z; \nu^2)\right|^2 dxdzd\nu \le K_1,$$

where K_1 is independent of t. Further, for the second integral in (19) we have

$$\iint_D \left|\int_0^1 \sin(\nu t)2\nu\sigma(\nu^2)\tau'_\nu(x, z; \nu^2)d\nu\right|^2 dxdz = \iint_D \left|\int_{\lambda_*}^{\lambda_{**}} \sin(\sqrt{\mu}t)\sigma(\mu)\tau'_\mu(x, z; \mu)2\sqrt{\mu}d\mu\right|^2 dxdz$$
$$\le (\lambda_{**} - \lambda_*)\iint_D \int_{\lambda_*}^{\lambda_{**}} \left|\sigma(\mu)\tau'_\mu(x, z; \mu)2\sqrt{\mu}\right|^2 d\mu dxdz = (\lambda_{**} - \lambda_*)\int_{\mathcal{D}_{**}} \int_{\lambda_*}^{\lambda_{**}} \left|\sigma(\mu)\tau'_\mu(x, z; \mu)2\sqrt{\mu}\right|^2 d\mu dxdz$$

$$+(\lambda_{**}-\lambda_{*})\int\limits_{D\setminus\overline{\mathcal{D}}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\sigma(\mu)\tau'_{\mu}(x,z;\mu)2\sqrt{\mu}\right|^{2}d\mu dxdz$$

The existence of the second term is evident, so we will consider the first one making use of the representation (13) of the function $\psi(x, z; \theta_1; \mu)$ in the domain $\mathcal{D}_{**}$:

$$\begin{aligned}
&\int\limits_{\mathcal{D}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\sigma(\mu)\tau'_{\mu}(x,z;\mu)2\sqrt{\mu}\right|^{2}d\mu dxdz\\
&\le 4\int\limits_{\mathcal{D}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\sigma(\mu)2\sqrt{\mu}\left(\frac{\sqrt{\mu}}{2\sqrt{1-\mu}}\right)'_{\mu}\right.\\
&\quad\left.\times\int\limits_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi(x',\mu)d\mu'\right|^{2}d\mu dxdz\\
&\quad+4\int\limits_{\mathcal{D}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\sigma(\mu)\left|\frac{\mu}{\sqrt{1-\mu}}\right.\\
&\quad\left.\times Q'_{\mu}(x,z;\mu)\varphi(Q(x,z;\mu),\mu)\right|^{2}d\mu dxdz\\
&\quad+4\int\limits_{\mathcal{D}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\sigma(\mu)\frac{\mu}{\sqrt{1-\mu}}\right.\\
&\quad\left.\times P'_{\mu}(x,z;\mu)\varphi(P(x,z;\mu),\mu)\right|^{2}d\mu dxdz\\
&+4\int\limits_{\mathcal{D}_{**}}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\sigma(\mu)\frac{\mu}{\sqrt{1-\mu}}\int\limits_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi'_{\mu}(\tilde{x},\mu)d\tilde{x}\right|^{2}d\mu dxdz.
\end{aligned}\tag{20}$$

We prove the existence of the last integral. For this purpose, it will be sufficient to prove the convergence of the series

$$\sum_{k=k_0}^{+\infty}\int\limits_{R_k}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\int\limits_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi'_{\mu}(\tilde{x},\mu)d\tilde{x}\right|^{2}d\mu dxdz,\tag{21}$$

where the integrals are taken over the trapezoids $R_k := \left\{(x,z)\,\middle|\,\frac{1}{l_1^{k+1}}<x<\frac{1}{l_1^{k}},\ 0<z<\alpha x\right\}$ and k_0 is sufficiently large. On transferring to the new variable of the integration $l = l(\mu)$ (see (16)) we have

$$\int\limits_{\lambda_{*}}^{\lambda_{**}}\left|\int\limits_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi'_{\mu}(\tilde{x},\mu)d\tilde{x}\right|^{2}d\mu\le K_2\int\limits_{l_1}^{l_2}\left|\int\limits_{P(x,z;l)}^{Q(x,z;l)}\tilde{\varphi}'_{l}(\tilde{x},l)d\tilde{x}\right|^{2}dl$$

$$\le K_2\int\limits_{l_1}^{l_2}\left|\tilde{P}(x,z;l)-\tilde{Q}(x,z;l)\right|\int\limits_{\tilde{P}(x,z;l)}^{\tilde{Q}(x,z;l)}\left|\tilde{\varphi}'_{l}(\tilde{x},l)\right|^{2}d\tilde{x}dl,$$

where K_2 is some positive constant only depending on λ_* and λ_{**}. Let

$$P_k := \max_{(x,z)\in R_k,\ l\in[l_1,l_2]}\tilde{P}(x,z;l),$$

$$Q_k := \min_{(x,z)\in R_k,\ l\in[l_1,l_2]}\tilde{Q}(x,z;l).$$

It is readily found that

$$P_k=\frac{l_2+1}{2l_1^{k}},\quad Q_k=\frac{l_2+1}{2l_1^{k+1}l_2},\tag{22}$$

therefore, for $(x, z) \in R_k$

$$\begin{aligned}
&\int\limits_{l_1}^{l_2}\left|\tilde{P}(x,z;l)-\tilde{Q}(x,z;l)\right|\int\limits_{\tilde{P}(x,z;l)}^{\tilde{Q}(x,z;l)}\left|\tilde{\varphi}'_{l}(\tilde{x},l)\right|^{2}d\tilde{x}dl\\
&\le\frac{K_3}{l_1^{k}}\int\limits_{Q_k}^{P_k}\left\{\int\limits_{l_1}^{l_2}\left|\varphi'_{l}(\tilde{x},l)\right|^{2}dl\right\}d\tilde{x},
\end{aligned}\tag{23}$$

where K_3 is some positive constant only depending on λ_*, and λ_{**}. In addition, it also follows from (22) that there exist natural numbers m_0 and r_0 that do not depend on k, such that for all k $l_1^{-(k+m_0)} \le Q_k < P_k \le l_1^{-(k-r_0)}$. Therefore, if the integrand in the right-hand side of inequality (23) is denoted by $\mathcal{F}$ by virtue of its nonnegativeness we have

$$\int\limits_{Q_k}^{P_k}\mathcal{F}d\tilde{x}\le\int\limits_{l_1^{-(k+m_0)}}^{l_1^{-(k+m_0-1)}}\mathcal{F}d\tilde{x}+\int\limits_{l_1^{-(k+m_0-1)}}^{l_1^{-(k+m_0-2)}}\mathcal{F}d\tilde{x}+\ldots+\int\limits_{l_1^{-(k-r_0+1)}}^{l_1^{-(k-r_0)}}\mathcal{F}d\tilde{x}.$$

Then

$$\begin{aligned}
&\sum_{k=k_0}^{+\infty}\int\limits_{R_k}\int\limits_{\lambda_{*}}^{\lambda_{**}}\int\left|\int\limits_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi'_{\mu}(\tilde{x},\mu)d\tilde{x}\right|^{2}d\mu dxdz\\
&\le(m_0+r_0+1)K_3\sum_{k=k_0}^{+\infty}\frac{S_k}{l_1^{k}}\int\limits_{l_1^{(k+1)}}^{l_1^{k}}\left\{\int\limits_{l_1}^{l_2}\left|\tilde{\varphi}'_{l}(\tilde{x},l)\right|^{2}dl\right\}d\tilde{x},
\end{aligned}\tag{24}$$

where S_k is the area of R_k.

Let $\tilde{x}\in[l_1^{-(k+1)}, l_1^{-k}]$. We designate by $l_{\tilde{x},i}$ the values l corresponding to the points of intersection between the straight line $x = \tilde{x}$ and the curves $x = \frac{1}{l^{k-i+1}}$, $i = 0, 1, 2, \ldots$ so that

$$\tilde{x} = \frac{1}{(l_{\tilde{x},0})^{k+1}} = \frac{1}{(l_{\tilde{x},1})^{k}} = \frac{1}{(l_{\tilde{x},2})^{k-1}}$$
$$= \ldots = \frac{1}{(l_{\tilde{x},i+1})^{k-i}} = \ldots,$$

and by $N(\tilde{x}, k)$ we denote the number of such intersections belonging to the segment $[l_1, l_2]$. It is readily found that for the sufficiently large k there holds $N(\tilde{x}, k) < \hat{\gamma} k$ where $0 < \hat{\gamma} < 1$ does not depend on k and $\tilde{x}$. At the same time, it is obvious that $l_{\tilde{x},0} \le l_1 \le l_{\tilde{x},1}$. We have

$$\int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} d\tilde{x} \int_{l_1}^{l_2} \left|\tilde{\varphi}'_l(\tilde{x}, l)\right|^2 dl$$
$$\le \int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \left\{ \sum_{i=0}^{N(\tilde{x},k)+1} \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} \left|\tilde{\varphi}'_l(\tilde{x}, l)\right|^2 dl \right\} d\tilde{x}.$$

We now use estimate (18) at $(n = 1)$. Then

$$\int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \left\{ \sum_{i=0}^{N(\tilde{x},k)+1} \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} \left|\tilde{\varphi}'_l(\tilde{x}, l)\right|^2 dl \right\} d\tilde{x}$$
$$\le \int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \left\{ \sum_{i=0}^{N(\tilde{x},k)+1} \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} C^2(1)(k-i)^2 l^{2(k-i)} dl \right\} d\tilde{x}$$
$$\le k^2 C^2(1) \int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \left\{ \sum_{i=0}^{N(\tilde{x},k)+1} (l_{\tilde{x},i+1})^{2(k-i)} (l_{\tilde{x},i+1} - l_{\tilde{x},i}) \right\} d\tilde{x}$$
$$\le k^2 C^2(1) \int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \{(N(\tilde{x},k)+2)(l_{\tilde{x},0})^{2(k+1)}\} d\tilde{x}$$
$$\le k^2 C^2(1) \int_{\Gamma_1^{(k+1)}}^{\Gamma_1^{k}} \{(\hat{\gamma}k+2) l_1^{2(k+1)}\} d\tilde{x} \le K_4 k^3 l_1^k,$$

where the positive constant K_4 is independent of k, hence the convergence of the series (24) is readily apparent.

Clearly, the same reasoning also shows the convergence of the first integral in the right-hand side of inequality (20) because the function $\varphi(x, \mu)$ increases no faster than $\varphi'_\mu(x, \mu)$.

Now we consider the second integral in the right-hand side of (20). Since $P'_\mu(x, z; \mu) = \frac{\alpha x - z}{2\alpha} l'_\mu$ it is obvious that

$$\iint_{\mathcal{D}_{**}} \int_{\lambda_*}^{\lambda_{**}} \left| \sigma(\mu) \frac{\mu}{\sqrt{1-\mu}} P'_\mu(x, z; \mu) \varphi(P(x, z; \mu), \mu) \right|^2 d\mu\, dx\, dz$$
$$\le K_5 \iint_{\mathcal{D}_{**}} |x| \int_{\lambda_*}^{\lambda_{**}} |\varphi(P(x, z; \mu), \mu)|^2 d\mu\, dx\, dz$$
$$\le K_5 \iint_{\mathcal{D}_{**}} |x| \int_{l_1}^{l_2} |\tilde{\varphi}(\tilde{P}(x, z; l), l)|^2 dl\, dz\, dz,$$

where K_5 and K_6 are some positive constants depending only on σ_0, λ_*, and λ_{**}. For every fixed point $(x, z) \in R_k$ we designate the values l corresponding to the points of intersection between the curve $x = \tilde{P}(x, z; l)$ with the curves $x_{k',j}(l)$ in the plane Olx through $l_{k,k',j}$. Let $N(x, z; k)$ be the number of such intersections $l_{k,k',j}$ belonging to the segment $[l_1, l_2]$, and let at the same time k' assume the values $\{k'_m, k'_m + 1, \ldots, k'_M\}$. Using the results of [1] we show the convergence of the series

$$\sum_{k=k_0}^{+\infty} \iint_{R_k} |x| \int_{l_1}^{l_2} |\tilde{\varphi}(\tilde{P}(x, z; l), l)|^2 dl\, dx\, dz, \qquad (25)$$

where k_0 is sufficiently large. For every point $(x, z) \in R_k$ we have

$$\int_{l_1}^{l_2} |\tilde{\varphi}(\tilde{P}(x, z; l), l)|^2 dl = \int_{l_1}^{l_{k,k'_M,n}} |\tilde{\varphi}(\tilde{P}(x, z; l), l)|^2 dl$$
$$+ \int_{l_{k,k'_M,n}}^{l_{k,k'_{M-1},n}} |\tilde{\varphi}(\tilde{P}(x, z; l)), l|^2 dl + \ldots$$
$$+ \int_{l_{k,k'_m,n}}^{l_2} |\tilde{\varphi}(\tilde{P}(x, z; l), l)|^2 dl$$
$$\le C(0) \left\{ \int_{l_{k,k'_M+1,n}}^{l_{k,k'_M,n}} l^{2k'_M} dl + \int_{l_{k,k'_M,n}}^{l_{k,k'_{M-1},n}} l^{2k'_M - 2} dl + \ldots \right.$$

$$+\int_{l_{k,k'_{m'},n}}^{l_{k,k'_{m}-1,n}} l^{2k'_m-2}dl\Bigg\}$$

$$\le C(0)(l_2-l_1)((l_{k,k'_M,n})^{2k'_M}+(l_{k,k'_M-1,n})^{2k'_M-2}+\dots$$

$$+(l_{k,k'_m-1,n})^{2k'_m-2})$$

$$= C(0)(l_2-l_1)(P(x,z;l_{k,k'_M,n})^{-2}$$

$$+P(x,z;l_{k,k'_M+-1,n})^{-2}+\dots+P(x,z;l_{k,k'_m-1,n})^{-2})$$

$$\le C(0)(l_2-l_1)(N(x,z;k)+1)\frac{4\alpha^2}{(\alpha x+z)^2}\le\frac{K_7k}{x^2},$$

where K_7 is independent of k, therefore, series (25) is really convergent.

Thus, at $n=1$ estimate (11) is proved. In order to prove that this estimate holds true also at $n=2$ it should be noted that

$$\tilde\psi(x,z;t)=-\frac{1}{t}\int_0^1\sin(\nu t)(2\nu\sigma(\nu^2))'_\nu\tau(x,z;\nu^2)d\nu$$

$$-\frac{1}{t}\int_0^1\sin(\nu t)2\nu\sigma(\nu^2)\tau'_\nu(x,z;\nu^2)d\nu$$

$$=-\frac{1}{t^2}\Bigg(\int_0^1\cos(\nu t)(2\nu\sigma(\nu^2))''_{\nu\nu}\tau(x,z;\nu^2)d\nu$$

$$+2\int_0^1\cos(\nu t)(2\nu\sigma(\nu^2))'_\nu\tau'_\nu(x,z;\nu^2)d\nu$$

$$+\int_0^1\cos(\nu t)2\nu\sigma(\nu^2)\tau''_{\nu\nu}(x,z;\nu^2)d\nu\Bigg).$$

The existence of the first two integrals in the last expression has been proved above and in order to prove the convergence of the third integral taking into account the representation $\psi(x,z;\theta_1;\mu)$ in the form of (13), it is obviously sufficient to prove the convergence of the series

$$\sum_{k=k_0}^{+\infty}\iint_{R_k}\int_{\lambda_*}^{\lambda_{**}}\left|\int_{P(x,z;\mu)}^{Q(x,z;\mu)}\varphi''_{\mu\mu}(\tilde x,\mu)d\tilde x\right|^2 d\mu dxdz. \qquad (26)$$

The proof of the convergence of this series is very similar to the proof of the convergence of the series (21) with the only difference consisting in the use of the estimate (18) for $n=2$. The cases $n=3, 4, \dots$ are considered in a similar way.

2. THE DISTRIBUTION OF THE INITIAL STATE ENERGY

It is readily apparent that for the solutions of the problem (1)–(4) there holds the law of energy conservation

$$\mathscr{E}(t,D):=\int_D(|u|^2+|v|^2+|w|^2)dxdz=\text{const}.$$

It is known that if the domain D is the supplement $\mathbf{R}^2\backslash\overline{\Omega}$ to some convex limited domain Ω, the energy of the initial perturbation is redistributed at $t\to\infty$ in such a way that its portion concentrated in any compact space $\mathscr{A}\in\mathbf{R}^2\backslash\overline{\Omega}$ tends to zero, which implies that the energy is dissipated. In our case, the situation is as follows: if in $\theta_1\in C_0^\infty[0,1]$, $\theta_2\in C_0^\infty[0,1/\alpha]$, $\sigma(\lambda)\in C_0^\infty[0,1]$, and the support $\operatorname{supp}\sigma$ is contained in the union of the intervals $(0,(1+\alpha^2)^{-1})$ and $((1+\alpha^2)^{-1},1)$ then for any $\varepsilon>0$ and $\delta>0$ there exists such $T=T(\varepsilon,\delta)$ that for the solution $U=(u,v,w)$ of the problem (1)–(4) for all $t>T$

$$\mathscr{E}(t,D_\varepsilon):=\iint_{D_\varepsilon}(|u|^2+|v|^2+|w|^2)dxdz<\delta, \qquad (27)$$

where $D_\varepsilon:=D\cap\{x>\varepsilon\}\cap\{z<1-\varepsilon\}$. Indeed, by assuming for simplicity that $\theta_2\equiv0$, $\operatorname{supp}\sigma\subset[\lambda_*,\lambda_{**}]\subset(0,(1+\alpha^2)^{-1})$, we have for the flow function ψ

$$\tilde\psi(x,z;t)=\int_0^1\cos(\sqrt{\lambda}t)\sigma(\lambda)\tau(x,z;\lambda)d\lambda$$

$$=-\frac{1}{t}\int_0^1\sin(\sqrt{\lambda}t)(2\sqrt{\lambda}\sigma(\lambda))'_\lambda\tau(x,z;\lambda)d\lambda$$

$$-\frac{1}{t}\int_0^1\sin(\sqrt{\lambda}t)\sigma(\lambda)\tau'_\lambda(x,z;\lambda)2\sqrt{\lambda}d\lambda$$

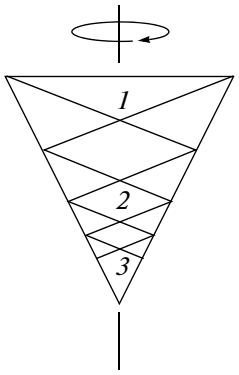
(see (19)). As has already been mentioned, in our conditions $\tau(x,z;\lambda)$ is indefinitely differentiable with respect to x, z, λ everywhere inside the prism $D\times(0,(1+\alpha^2)^{-1})$ and can be continuously extended with all of its derivatives to the entire surface of the prism $(D\cap\{x>\varepsilon\})\times(0,(1+\alpha^2)^{-1})$, $\varepsilon>0$. We have

$$u=\frac{1}{t}\int_0^1\sin(\sqrt{\lambda}t)(2\sqrt{\lambda}\sigma(\lambda))'_\lambda\tau'_z(x,z;\lambda)d\lambda$$

$$+\frac{1}{t}\int_0^1\sin(\sqrt{\lambda}t)\sigma(\lambda)\tau''_{z\lambda}(x,z;\lambda)2\sqrt{\lambda}d\lambda,$$

therefore,

$$\iint_{D\cap\{x>\varepsilon\}}|u|^2dxdz$$



Fluid motion in a conic container.

$$\leq \frac{2}{t^2}\int\limits_{D\cap\{x>\varepsilon\}}\int\left|\int\limits_0^1 \sin(\sqrt{\lambda}t)(2\sqrt{\lambda}\sigma(\lambda))'_\lambda \tau'_z(x, z; \lambda)d\lambda\right|^2 dxdz$$

$$+\frac{2}{t^2}\int\limits_{D\cap\{x>\varepsilon\}}\int\left|\int\limits_0^1 \sin(\sqrt{\lambda}t)\sigma(\lambda)\tau''_{z\lambda}(x, z; \lambda)2\sqrt{\lambda}d\lambda\right|^2 dxdz$$

$$\leq \frac{2}{t^2}\int\limits_{D\cap\{x>\varepsilon\}}\int\int\limits_0^1 \left|(2\sqrt{\lambda}\sigma(\lambda))'_\lambda \tau'_z(x, z; \lambda)\right|^2 d\lambda dxdz$$

$$+\frac{2}{t^2}\int\limits_{D\cap\{x>\varepsilon\}}\int\int\limits_0^1 \left|\sigma(\lambda)\tau''_{z\lambda}(x, z; \lambda)2\sqrt{\lambda}\right|^2 d\lambda dxdz$$

$$\leq \frac{K_\varepsilon}{t^2} \longrightarrow 0 \quad (t \longrightarrow \infty),$$

where the positive constant K_ε is independent of t. The remaining terms in (27) are estimated similarly, and, therefore, the proof of this inequality has been completed.

It follows from it that for such a choice of initial perturbations $\mathbf{U}_0 = (u_0, v_0, w_0)$ the solutions of the problem (1)–(4) are such that the entire energy of their initial state with time appears to be almost completely concentrated in the arbitrarily small neighborhoods of the vertices O and B of the domain D. It is at the same time obvious that if $\theta_2 \equiv 0$ or $\operatorname{supp}\sigma \in (0, (1 + \alpha^2)^{-1})$, the energy is only accumulated in the vicinity of the point O; accordingly, if $\theta_1 \equiv 0$ or $\operatorname{supp}\sigma \in ((1 + \alpha^2)^{-1})$, the energy is accumulated in the neighborhood of the point B.

According to [1], the suggested method for constructing the exact solutions of the problem (1)–(4) can be used for a certain rather wide class of domains with angular points, e.g., when D is a "curvilinear triangle," in which its sides OA and OB are some smooth curves intersecting at the point O at a nonzero angle and such that for all λ from a certain interval (λ', λ'') the beams of the characteristic directions, whose law of reflection from the boundaries is described in Section l, are localized in the angle with the vertex O. Clearly, the behavior of these solutions at $t \longrightarrow \infty$ will be analogous to that described above.

The performed study of our two-dimensional model problem in an infinite cylinder will make it possible to forecast the behavior of an ideal fluid both inside the cone and inside, say, a torus symmetrical about the turning axis and such that its axial section is a triangle D, i.e., it can be affirmed that at a certain selection of the initial perturbations the fluid oscillations will be such that almost all the energy of these perturbations in the course of time will prove to be in an arbitrarily small neighborhood of some edge or cone point.

In conclusion, we discuss the experimental studies. As is known, researchers have taken a lively interest in the problem of rotating fluid behavior in containers with different configurations for a long time and the following experiments were carried out. A right circular cone fully filled with fluid was placed on a table rotating about a vertical axis with angular velocity $\mathbf{\Omega} = \Omega(1 + \varepsilon\cos\omega t)\mathbf{k}$ (see [7]).

This setup was analogous to that earlier used in [8] in order to investigate inertial modes in a sphere. At the same time, while in the case of a sphere undergoing slow changes in ω the fluid passed through some states of resonance corresponding to the eigenfrequencies of the axially symmetric inertial waves, in the cone the picture was totally different, and here only a qualitative description of the fluid behavior was obtained.

The figure shows a steady internal flow as it was seen due to the lighting of aluminum flakes introduced in the fluid. It appears that the amplitude of the fluid oscillation along the axis increases from the near-zero value in domain 1 to a value approaching the amplitude of superimposed oscillations in domain 2.

However, in the immediate vicinity of the vertex (in domain 3) the amplitude of the oscillations becomes very small again. At the same time, this kind of a flow remains the same for the values of ω, under which the characteristics corresponding to them are localized in the vertex of the cone.

Such fluid behavior did not receive the proper theoretical explanation at that time. Our study of the solutions for the two-dimensional model problem offers just such an explanation of the fluid behavior, i.e., there cannot be any resonance phenomena in this situation because the spectrum of the respective operator is continuous and the amplitude of the fluid motion increases as it is approaching the vertex because the energy with time appears to be concentrated in the vicinity of the vertex if the values of ω are such that the characteristics of the considered hyperbolic equation under the reflection from the boundary are localized just in the angle corresponding to this vertex. On the other hand, it appears that in the investigations of the oscillation of the real rotating fluid in a

container with such boundary specifics one must not neglect the viscosity and, therefore, the calculations must allow for the corresponding nonlinear systems.

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