ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 6, pp. 454–459. © Allerton Press, Inc., 2010.
Original Russian Text © N.A Vinnichenko, A.V. Uvarov, 2010, published in Vestnik Moskovskogo Universiteta. Fizika, 2010, No. 6, pp. 28–32.

Improvement of the Oberbeck–Boussinesq Approximation for Solving Unstationary Convection Problems in a Closed Cavity

N. A Vinnichenko and A. V. Uvarov

Department of Molecular Physics, Faculty of Physics, Moscow State University, Moscow, 119991 Russia
e-mail: nickvinn@yandex.ru, uvarov@phys.msu.ru

Received: February 17, 2010

Abstract—The validity of using simplified models of compressible fluids for calculations of unstationary convection flows inside closed cavities is tested for the problem of hot gas cooling in a rectangular cavity with cold walls. The Oberbeck–Boussinesq approximation and other simplified models are shown to yield incorrect values of pressure, density, and temperature if average pressure change is not negligible. Modification of the numerical method is proposed, which allows one to describe the temporal dependence of pressure correctly without loss of computational efficiency. The results obtained by using a modified Oberbeck–Boussinesq approximation and the complete equations for a fully compressible fluid are compared.

Keywords: unstationary convection, Oberbeck–Boussinesq approximation.

DOI: 10.3103/S0027134910060056

INTRODUCTION

The Oberbeck–Boussinesq approximation (OB) [1] has been a very productive tool for the analysis of convective stability. Despite the limitations of typical sizes of the system and the gradients of parameters [2–3], this approach can significantly simplify a system of hydrodynamic equations and even in some cases analytical analysis. The development of computational methods only strengthened the role of this approach; because of the large difference in the time propagation of sound waves and the typical times associated with the convective motion of the medium, the solution of the complete system of equations involves a considerable amount of time. Indeed, for the simulation of a convective flow with velocity $V_0 \approx 10^{-2}$ m/s it is necessary to calculate it, at least at the time L/V_0, where L is the typical size of the system. Under the restriction imposed on the time step for the explicit scheme $dt \leq drL/c_s$, where c_s is the velocity of sound and dr is the dimensionless grid step, this is $c_s/(V_0 dr)$ time steps. Thus, for a typical grid containing about 200 nodes in each coordinate, this estimate gives 7×10^6 steps of time. The use of OB approximation and other simple models that take the compressibility of the medium into account allows the use of more efficient computational methods and the increase in the time step is about 500 times, respectively, reducing the computation time. Of course, with a significant increase in the Rayleigh number, even in standard problems of the heat flow passing through the volume, the OB approximation becomes inapplicable (see, for example, [4], where comparison with an exact two-dimensional solution for the problem of heating and heat removal through side walls showed that for Rayleigh numbers $\mathrm{Ra} \geq 3 \times 10^8$ deviations from the OB approximation become important), but for Rayleigh numbers that are not very large, this method is still a very important, practical, and very widely used approximation.

At the same time there is a class of problems where the OB approximation is formally invalid, even when at $\mathrm{Ra} \ll 10^8$. Such a situation arises in the calculation of unstationary convection in closed systems, such as in a detector of leaks in pipes [5]. Leak detection is based on measuring the pressure difference in two identical cells, measurement and control, in which gas is supplied from the main pipe. Because a gas noticeably heats up during the process of gas recruitment into a chamber, then immediately after the closing of the intake valve it is cooled by heat exchange with the walls of the chamber, a convective flow occurs. While the temperature of gas is not stable its pressure also changes. This defines the minimum time that must elapse from the recruitment of gas in the chamber until the moment of pressure difference measurement so that the measurement is sufficiently accurate and reflects the difference between the amount of gas in the cells, rather than the temperature difference. To find this time [5] used a simplified approach to the introduction of the effective heat transfer coefficient,

which was determined experimentally. Modern computational methods allow us to solve this problem exactly, taking gas dynamics equations into account. Next, for an example of this problem, which is, on the one hand, important in terms of technical applications and on the other hand is the classical problem of convection in a closed cavity, it will be shown that the OB approximation does not describe the dependence of pressure, density, and temperature from the time when the change in the average pressure (over space) cannot be ignored. A similar situation arises in problems in which the pressure within the system grows during volumetric heating of a closed volume.

We propose a simple way to improve the OB system, which allows one to retain all the advantages of the method and at the same time greatly improve calculation accuracy.

THE OB APPROXIMATION AND A SIMPLIFIED MODEL OF A COMPRESSIBLE FLUID

Consider a complete system of gas dynamics equations with account of viscosity and thermal conductivity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0,$$

$$\rho \frac{d\mathbf{v}}{dt} = -\mathrm{grad} p + \eta \Delta \mathbf{v} + \frac{\eta}{3}\mathrm{grad}\,\mathrm{div}\mathbf{v} + \rho \mathbf{g},$$

$$c_p \rho \frac{dT}{dt} - \frac{dp}{dt} = \lambda \Delta T \qquad (1)$$

$$+ \frac{\eta}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_k} + \frac{\partial \mathrm{v}_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial \mathrm{v}_l}{\partial x_l}\right)^2 + \rho \mathbf{g}\mathbf{v},$$

$$\rho = \frac{p\mu}{RT},$$

where ρ, $\mathbf{v}$, p, T are the density, velocity, pressure, and temperature, respectively, η and λ are the viscosity and thermal conductivity coefficients, μ and c_p are the molar mass of a gas and heat at a constant pressure, R is the gas constant, g is the acceleration of gravity, δ_{ik} is the Kronecker symbol, and $d/dt = \partial/\partial t + (\mathbf{v}\nabla)$.

In the OB approximation it is assumed that pressure changes and associated changes in density can be neglected in all the equations of system (1), except for the equations of motion. In addition, since the velocity of convective flows is small compared with the velocity of sound, we ignore the third term on the right side of the equation of motion and the second on the right side of the energy equation. The work of gravity $\rho \mathbf{g}\mathbf{v}$ can be ignored if $g L/(c_p\theta) \ll 1$, where L is the typical size of the system, and θ is the typical temperature difference. Thus, we obtain the equations of gas dynamics in the OB approximation:

$$\mathrm{div}\mathbf{v} = 0,$$

$$\rho \frac{d\mathbf{v}}{dt} = -\mathrm{grad} p + \eta \Delta \mathbf{v} + \rho \mathbf{g},$$

$$c_p \rho \frac{dT}{dt} = \lambda \Delta T, \qquad (2)$$

$$\rho = \frac{p_0 \mu}{RT},$$

where p_0 is the pressure in the undisturbed environment.

This approximation is usually used for calculations of convective flows, because it does not account for the propagation of sound waves, but accurately describes the result, viz., mass loss from the heated areas, provided that the pressure remains approximately constant and equal (up to a hydrodynamic pressure difference $\rho v^2 \ll p_0$) in the entire area. The value of a time step during the solution of equations (2) is not determined by the velocity of propagation of sound waves c_s as in the case of the system (1), but by the velocity of convective flow $v_0 \ll c_s$.

In the case of unstationary convection in a closed cavity the spatial pressure gradients are small, but the change in pressure over time cannot be neglected, so the direct application of OB approximation leads to erroneous results. Indeed, using the equation of state in the form $\rho = p_0\mu/(RT)$, for example, in the problem of a cooling gas in a cavity with cold walls, leads to an unphysical increase in the mass of gas contained in the cavity. The OB approximation assumes an "average" isobaric process, while cooling or heating gas in a closed cavity is "on average," an isochoric process. It is easy to estimate the error change of mass δm during the description with the help of the OB approximation of evolution of gas, which received the amount of heat Q:

$$p_0 V = \frac{m}{\mu} R T_1,$$

$$p_0 V = \frac{m - \delta m}{\mu} R T_2,$$

$$Q = c_v m (T_2 - T_1),$$

where V is the volume of the cavity, m is mass of the gas, T_1 and T_2 are the initial and final temperatures of the gas, and c_v is the heat capacity at constant volume. From the state equations and thermal balance recorded it follows that $\delta m = m(T_2 - T_1)/T_2 = Q/(c_v T_2)$. Note that the relative error does not depend on the size of the cavity. Thus, even for a large but closed cavity, application of the OB approximation leads to erroneous results.

Violation of the conservation of mass in the calculation of unstationary convection in a closed cavity is typical also for more accurate models, which neglect the total derivative of pressure over time in the energy equation, for example, the simplified model of a compressible fluid (SMCF):

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0,$$

$$\rho \frac{d\mathbf{v}}{dt} = -\mathrm{grad} p + \eta \Delta \mathbf{v} + \frac{\eta}{3} \mathrm{grad}\,\mathrm{div}\mathbf{v} + \rho \mathbf{g}, \quad (3)$$

$$c_p \rho \frac{dT}{dt} = \lambda \Delta T + \frac{\eta}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_k} + \frac{\partial \mathrm{v}_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial \mathrm{v}_l}{\partial x_l}\right)^2 + \rho \mathbf{g}\mathbf{v},$$

$$\rho = \frac{p\mu}{RT}.$$

In solving the system of equations (3) at each time step, first the temperature is determined from the energy equation. Then from the equation of state a density estimate for a new step is determined, which is used to approximate the time derivative in the continuity equation. This approach allows us not to describe the propagation of sound waves and use all the benefits of OB approximations, while taking the part of effects that is related to the compressibility of a gas into account. SMCF gives results very similar to those obtained using the complete system of equations to describe flows associated with the heat emission in unrestricted areas, for example during calculation of the effect of relaxation on the structure of the Karman vortex street in a vibrationally nonequilibrium gas [6]. For a closed area, where the pressure varies considerably during heating or cooling, the neglect of a term $\partial p/\partial t$ in the energy equation (term $(\mathbf{v}\nabla)p$ is negligible) leads to the same erroneous results as in the case of OB approximation.

It should also be noted that not only are the density and pressure distributions obtained in the calculation of convection in a closed cavity within the framework of simplified models invalid, but the distribution of temperature, and hence the velocity field of the convective flow are incorrect as well. Indeed, on "average" the isochoric process of gas temperature changes in a closed cavity is not described by the energy equation of $c_p \rho dT/dt = \lambda \Delta T$ taken in the OB approximation.

MODIFICATION OF SIMPLIFIED MODELS

Above it was shown that the OB and SMCF approximations lead to erroneous results when calculating unstationary convection in a closed cavity, namely: (1) the mass of the gas enclosed in the cavity is not conserved, (2) the energy equation $c_p \rho dT/dt = \lambda \Delta T$ fails to describe correctly the temperature change under constant volume. Accordingly, is necessary to modify this approach so that during the calculation of flows with low Mach numbers sufficiently precise values will be obtained. It is necessary to preserve the overall structure of the equations and numerical algorithm, since this defines the permissible value of the time step and the efficiency of calculation.

We assume that the pressure in the cavity is approximately the same at all points ($\rho v^2 \ll p_0$), but depends on time. That is, the spatial gradients of pressure and associated changes in density, as in the OB approximation, can be ignored, but the partial derivative of pressure over time cannot be neglected. If we also take the fact that during an isochoric process the average pressure $p \sim T$ and $\partial p/\partial t \approx (R\rho/\mu)\partial T/\partial t$ into account, then the left side of the energy equation of system (1) becomes:

$$c_p \rho \frac{dT}{dt} - \frac{dp}{dt} \approx c_p \rho \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} + c_p \rho (\mathbf{v}\nabla) T \approx c_v \rho \frac{\partial T}{\partial t} + c_p \rho (\mathbf{v}\nabla) T. \quad (4)$$

Thus, the energy equation, as in the OB and SMCF approximations, can be solved for the temperature, but this takes changes in the average pressure over time into account.

To find corrections to the density and pressure ensuring the conservation of mass, we represent density and pressure in the form $p(\mathbf{r}, t) = p_1(\mathbf{r}, t) + \delta p(t)$, $\rho(\mathbf{r}, t) = \rho_1(\mathbf{r}, t) + \delta\rho(t)$, where $p_1(\mathbf{r}, t)$ and $\rho_1(\mathbf{r}, t)$ satisfy the equations of the OB approximation (or, if we more fully consider the effects of gas compressibility, SMCF), and $\delta p(0) = \delta\rho(0) = 0$. For the law of conservation of mass and the equation of state hold while accounting for $\delta p(t)$ and $\delta\rho(t)$ it is necessary that

$$\int (\rho_1(\mathbf{r}, t) + \delta\rho(t)) dV = \int \rho_{\mathrm{init}}(\mathbf{r}) dV,$$

$$\int \delta\rho(t) dV = \int \frac{\mu \delta p(t)}{RT(\mathbf{r}, t)} dV,$$

where $\rho_{\mathrm{init}}(\mathbf{r})$ is the density distribution at the initial time.

We obtain:

$$\delta\rho(t) = \frac{\int (\rho_{\mathrm{init}}(\mathbf{r}) - \rho_1(\mathbf{r}, t)) dV}{\int dV},$$

$$\delta p(t) = \frac{R}{\mu} \frac{\int (\rho_{\mathrm{init}}(\mathbf{r}) - \rho_1(\mathbf{r}, t)) dV}{\int \frac{dV}{T(\mathbf{r}, t)}}. \quad (5)$$

Thus, the proposed modification of the OB and SMCF approximations during the calculation of unstationary convection in a closed cavity is (1) to use the left part of the energy equation of the form (4) and (2) to add to the density and pressure distribution found at the end of each time step of the corrections given by (5).

PROBLEM STATEMENT AND NUMERICAL METHODS

To check the accuracy of the simplified models two-dimensional numerical simulation of cooling gas in a square cavity with cold walls was carried out, corresponding to the simplified description of the processes occurring in the cells of a leak detector [5]. To solve the complete system of equations (1) we used the

Godunov method of the second order of accuracy based on the Rodionov scheme [7] with the exact solution of the Reimann problem and a van Leer TVD limiter. The solution of the equations of the OB approximations (2) and SMCF (3) was performed using the semi-implicit scheme [6, 8]. Time integration was carried out using the third order Runge–Kutta method and the Crank–Nicholson scheme for the dissipative terms. During the approximation of the spatial derivatives we used central differences. For the simultaneous solution of the equations of continuity and motion in the case of simplified models the projection method for determining the pressure was applied.

The following problem parameters were given: a constant wall temperature $T_{\text{wall}} = 300$ K, maximum initial gas temperature $T_{\max} = 360$ K, side cavity $L = 5.2$ mm, initial gas pressure $p_{\min} = 5$ atm, and all the parameters of the medium were consistent with air. The length of the cavity side was specifically reduced by a factor of 10 compared with the parameters of the leakage detector [5] to reduce the Rayleigh number and the turbulent convection that takes place in an experiment, go to laminar. Thus, the question of the accuracy of the equations used is separate from the problem of the simulation of turbulence in a real flow. The following boundary conditions were set on the cavity walls: $\mathbf{v} = 0$, $T = T_{\text{wall}}$; in the case where the full model (1) required an additional condition for the pressure $\partial p/\partial n = 0$. Since, in contrast to the Godunov method, a semi-implicit scheme [6, 8] loses its stability in areas with large gradients of the parameters, the initial conditions in all cases were given in the form:

$$\begin{aligned} &\mathbf{v} \equiv 0, \quad T(x, y) = T_{\max} \\ &-(T_{\max} - T_{\text{wall}})\exp(-\delta(x, y)/d), \\ &p \equiv p_{\text{init}}, \quad \rho(x, y) = p\mu/(RT(x, y)), \end{aligned} \tag{6}$$

where $\delta(x, y)$ is the distance to the nearest wall and $d = 0.2L$ is the thickness of transition layer. The simulation was performed for grids of 100×100, 200×200, and 300×300 nodes and for simplified models for the grids of 50×50, 100×100, and 170×170 nodes for the complete system of equations (1). The grid spacing in all cases decreased when approaching the cavity walls for a more accurate description of the areas with large temperature gradients. The semi-implicit scheme [6, 8] is characterized by significantly lower numerical dissipation compared with the Godunov method, so its results obtained using different grids have only minor differences. For the calculation of different grids by the Godunov method, in contrast, there are significant differences: the velocity of the convective flow in the calculation on the coarse grid is much smaller due to the influence of numerical dissipation. Even when based on the most detailed of the grids, 170×170 nodes, with a minimum dimensionless step of 2.5×10^{-3} this effect was more noticeable, but a further increase in the number of nodes is impossible because of calculation time increases (calculation of the complete system of equations (1) takes more than 3 weeks on a personal computer).

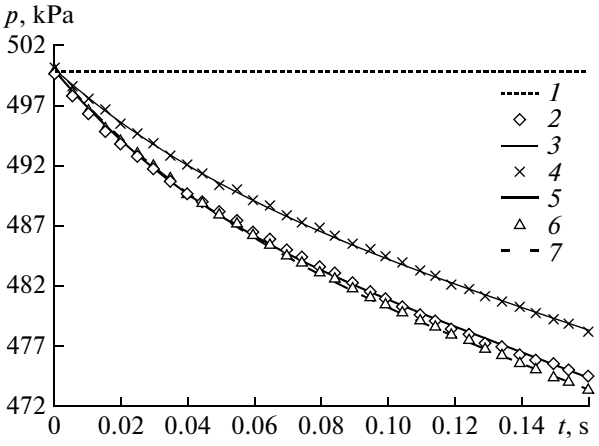

Fig. 1. Dependence of pressure in the center of the cavity from the time. *1*, SMCF without modifications; *2*, complete system (1), grid 50×50; 3, complete system (1), grid 170×170; *4*, the OB approximation taking (5) into account but excluding (4), *5*, SMCF taking (5) into account but excluding (4); *6*, the OB approximation with account for (4) and (5); *7*, SMCF with account for (4) and (5).

RESULTS OF NUMERICAL SIMULATION

Figure 1 shows the changes in pressure at the center of the cavity over time for different models and grids. The OB and SMCF approximations without modification (curve *1*) do not describe the change in pressure; the mean pressure in the cavity remains constant due to a non-physical increase in the mass of the gas. Curves *2* and *3*, which were obtained using different grids for the complete system of equations (1), are virtually identical, since the magnitude of the numerical dissipation has little influence on this dependence. The maximum deviation from the complete model of the curves obtained for the approximations of OB and SMCF considering the proposed modifications (curves *6* and *7*), amount to 3% of the observed changes in pressure and can be explained by the approximate nature of the equations and differences in the numerical methods and grids. Evaluation of the influence of the left side of the energy equation in the form of (4) can be performed using the deviation of the curves for the OB and SMCF models with correction of the density and pressure at each time step, but without modifying the energy equation (curves *4* and *5*). This deviation is 9%, which confirms the need for this modification in order to accurately describe the rate of heat transfer. Since the Mach number of the resulting convective flow is very low ($M \approx 6 \times 10^{-5}$) the results obtained by using OB and SMCF approximations are virtually identical.

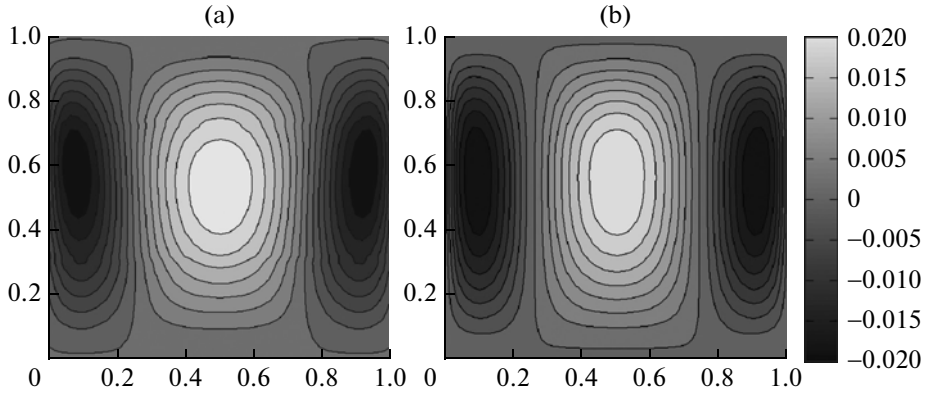

Fig. 2. Distribution of vertical velocity component (m/s) at the moment $t = 0.14$ sec. a, SMCF taking the modifications into account; b, a complete system (1), a 170×170 grid.

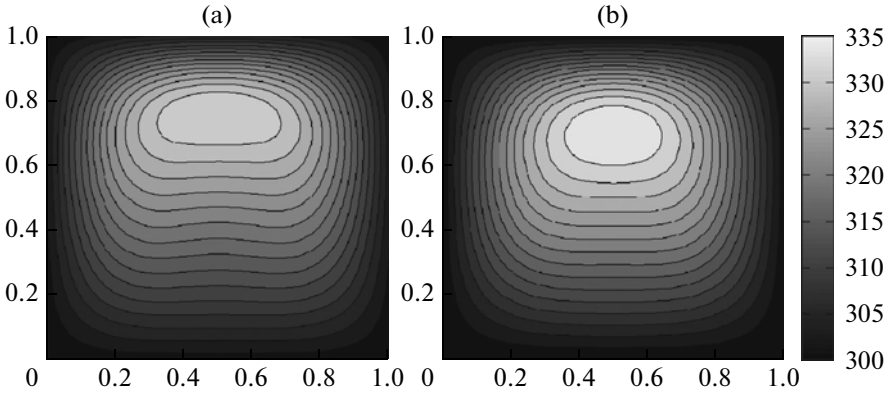

Fig. 3. The temperature distribution (K) at the moment of time $t = 0.14$ sec. a, SMCF taking the modifications into account; b, the complete system (1), the 170×170 grid.

Figures 2 and 3 show the calculated field of vertical velocity and temperature for the time $t = 0.14$ sec. when the rate of convective flow reaches its maximum value. The results obtained using the modified simplified model of a compressible fluid and the solution of the complete system (1) are very similar. The difference between the maximum temperature (333.3 K for complete model and 330.7 K for SMCF) is 4.3% of initial temperature difference between the gas and walls of the cavity. Somewhat smaller value of the maximum vertical velocity for the complete model (0.0178 m/c) compared with the SMCF model (0.0206 m/s) are probably due to the influence of numerical dissipation in the solution of the complete system.

CONCLUSIONS

The use of simplified models of the compressibility of a medium in the case of slow convection currents eliminates the need to describe the rapid propagation of sound waves, which significantly (by 500 times) reduces computation time. It was shown that the use of simplified models, which neglect the time derivative of pressure in the energy equation, leads to erroneous results in the calculation of unstationary convection in a closed cavity, when the change in average pressure cannot be ignored. In particular, we showed that for the OB and SMCF approximations mass of the gas in the cavity was not conserved and the rate of heat transfer was not described accurately. A modification of these approximations, which eliminates these shortcomings, was proposed. We performed a numerical simulation of a cooling gas in a square cavity with cold walls in various models. The differences between the results obtained for simplified models with the proposed modifications from the results for the complete system of equations are 3–4%, which is within the error of the numerical methods we used.

ACKNOWLEDGMENTS

This study was supported in part by RFBR (grant 09-08-00961-a).

REFERENCES

1. G. Z. Gershuni and E. M. Zhukhovitskii, *Convective Instability of Incompressible Fluids* (Keter, Jerusalem, 1976; Nauka, Moscow, 1972).
2. A. V. Uvarov and A. I. Osipov, Vestn. Mosk. Univ., Fiz. Astron., No. 5, 42 (1998).
3. R. Kh. Zeytounian, Comptes Rendus Mecan. **331**, 575 (2003).
4. H. Mlaouah, T. Tsuji, and Y. Nagano, Int. J. Heat Fluid Flow **18**, 100 (1997).
5. L. G. Harus, M. Cai, K. Kawashima, and T. Kagawa, Meas. Sci. Technol. **17**, 411 (2006).
6. A. I. Osipov, A. V. Uvarov, and N. A. Vinnichenko, Phys. Fluids **18**, 105106 (2006).
7. A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov, *Mathematical Aspects of Numerical Solution of Hyperbolic Systems* (Fizmatlit, Moscow, 2001) [in Russian].
8. J. Kim, D. Kim, and H. Choi, J. Comput. Phys. **171**, 132 (2001).