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The Dependence of the Regularity of an Output Pulse Signal of a Neural Model on the Frequency Ratio of Input Oscillations

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Abstract—A model of a neural system generating a train of delta pulses under the effect of noises and sinusoidal signals with commensurable frequencies is considered. The dependence of the regularity of the above pulse train on the ratio of input oscillation frequencies is obtained using the theory of the hidden Markov chain.

Keywords: conditional Markov process, neuron, spike, theory of information, entropy, regularity.

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INTRODUCTION

The theory of information makes it possible to study very urgent engineering problems that are mainly related to data transfer, storage, and acquisition [1]. Based on this theory, encoding and decoding algorithms are developed. These allow to efficiently transmit signals through noisy channels and to reveal and distinguish useful informational content in complex excess processes. A vivid example of a system receiving complex signals and possessing noisy channels [2] and encoding–decoding subsystems is the mammal brain and, in particular, the auditory neural system considered in the present study [3]. For it, a stochastic mathematical model using signals that are sinusoidal in time with commensurable frequencies Ω_1 and Ω_2 ($\Omega_1/\Omega_2 = m/n$, where $m, n \in \mathbb{N}$) as an external deterministic action, was constructed.

The aim of the study is to reveal the relationship between the m/n ratio and regularity of an output pulse signal at a constant noise level in the system. This problem has been studied earlier [4] from a probabilistic point of view: distributions of intervals between pulses at the output of a neural system showing that the larger m and n, the less regular an output signal is (and vice versa) were constructed. In the present paper, a more obvious quantitative description of signal regularity and its behavior in case of variations in the frequency ratio of a pair of input oscillations is given based on the apparatus of the theory of information.

The necessary data on the studied model and some results of its analysis published in previous papers are given below in brief. Then, the possibility of applying these results to determine the specific informational entropy of an output signal is substantiated. A formal definition of a signal "regularity" is introduced in the third section, and its dependence on the frequency ratio Ω_1/Ω_2 is analyzed. A comparison of this dependence with a rough numerical estimate is given in section four, followed by our conclusions.

1. A MODEL OF AN AUDITORY ANALYZER

The auditory analyzer of mammals is made so that a sound signal decomposes into frequency components, each of which acts on a corresponding group of sensory neurons (sensors). These generate trains of electric pulses (spikes) that are sent along nerve fibers to the brain for further processing [3]. A model of the studied system consists of two sensors and one intermediate neuron (an interneuron IN) simulated by the stochastic Langevin equations in the form of a so-called "leaky integrate-and-fire neuron" [5]. A sum of spike trains (ST) generated by unconnected sensors under the effect of sinusoidal signals and statistically independent noises is the input signal for an IN which, in its turn, is also affected by an independent source of noise (Fig. 1). A detailed description of the mathematical model, the values of the analyzed parameters, the methods and the results of the analysis are given in detail in [4, 6, 7].

Let us remember that an ST at the IN output is considered as a point process, since all spikes are considered identical. This random process is entirely characterized by interspike intervals (ISI) and, generally speaking, is not Markov. However, it is possible to construct a hidden Markov model for it [8–10] which, in the case where m and n are integers, has $M = m + n - 1$ states and a corresponding transition matrix with the elements $\pi_{i,j}$, where $i, j = 0, 1, ..., M - 1$.

The study of such an elementary model is topical from the point of view of the theory of music perception [11, 12]. It is known that the simultaneous sounding (accord) of two sinusoidal signals (tones) with a

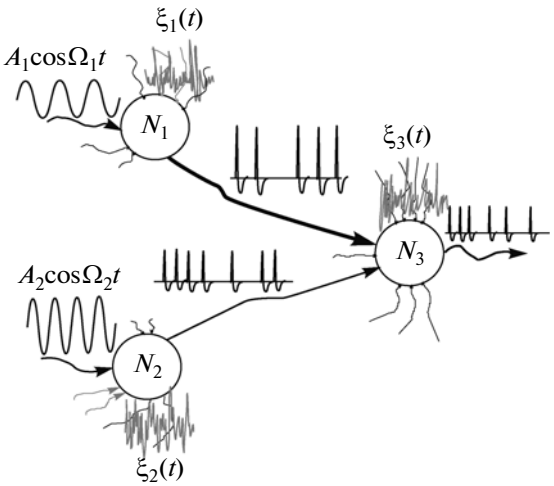

Fig. 1. The analyzed system of three neurons. N_1 and N_2 are sensors; N_3 is an interneuron; $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$ are statistically independent white Gaussian noises.

frequency ratio expressed by *small* integers m and n (2/1, 3/2. etc.) yields a pleasant harmonic perception and is called *consonance*. If m and n are "large" integers (e.g., 16/15, 45/32), a pair of such tones yields *dissonance*. Thus, the obtained results have a direct field of application. In addition, the use of musical terminology, as well as frequency ratios Ω_1/Ω_2 corresponding to musical intervals turns out to be very convenient in the described studies.

2. SPECIFIC ENTROPY OF A MARKOV CHAIN

Within the hypothesis of time data encoding [11], a neuron is considered to generate a finite or a countable set of ISIs, whose combinations are some messages. In the analyzed case, an ISI train is, on average, unambiguously determined by a sequence of states of a hidden Markov chain. We say "on average," since a transition from one state to another yields generation of an ISI, the length of which, generally speaking, has some probabilistic distribution near the mean value corresponding to the transition. It is assumed that useful information is encoded by this very set of well-distinguishable ISIs. Their fluctuations not being taken into account are well-described near the mean values by a narrow normal distribution [13]. Let us remember that the area under the density of ISI distribution near the mean value corresponding to a transition from the i-th state to the j-th one determines the probability of such a transition, i.e., an element of a transition matrix $\pi_{i,j}$ [7]. As a result, the known equation for the specific informational entropy [1] can be used without analyzing additional characteristics

$$H = -\sum_{i=0}^{M-1} p_i \sum_{j=0}^{M-1} \pi_{ij}\log_2\pi_{ij}, \qquad (1)$$

where p_i is the limiting probability of the i-th state of a hidden Markov chain [14].

3. FAREY SEQUENCE AND THE MAIN RESULT

An acceptable solution to the problem formulated in the Introduction is likely to be the dependence of regularity (an expression of a regularity through the specific informational entropy is introduced below) of the output ST on the frequency Ω_1 varied within an octave: $\Omega_2 < \Omega_1 < 2\Omega_2$, with the frequency Ω_2 being fixed. However, in this case we face the problem of selecting some values of Ω_1 for which $\pi_{i,j}$ and p_i are determined using numerical methods. According to the above-said, Eq. (1) is applicable for the analyzed model in the case where Ω_1/Ω_2 is a rational number. Thus, all that is left to do is to supplement the set of relationships of a natural musical sequence, which was studied earlier in [4], by intermediate elements. It is proposed to do this using the *Farey sequence* that orders rational ratios in a set of real numbers.

A Farey sequence F_r of the order r is an ascending sequence of nonnegative irreducible fractions which do not exceed 1 with denominators not larger than r [15]. For example

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\},\ F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{2}\right\},\ F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}.$$

To derive F_{r+1} from F_r, the last sequence is copied as a whole and each pair of neighboring ratios α/β and γ/δ is considered. If $\beta + \delta \le r + 1$, then $(\alpha + \gamma)/(\beta + \delta)$ is inserted between these ratios:

$$F_r = \left\{\frac{0}{1}, \ldots, \frac{\alpha}{\beta}, \frac{\gamma}{\delta}, \ldots, \frac{1}{2}\right\} \Rightarrow F_{r+1}$$

$$= \left\{\frac{0}{1}, \ldots, \frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}, \ldots, \frac{1}{2}\right\}.$$

The Farey sequence makes it possible to approximate irrational numbers to rational ones with the required accuracy. This term is mentioned in the theory of coupled dynamic systems, since it characterizes a ratio of the frequencies of a generator pair near which synchronization is observed. In the present study, for the addition of intermediate elements to a natural musical sequence to be convenient we consider the ratios from 1/1 (a unison) to 2/1 (an octave), i.e., the right half of the Farey sequence with "inverted" fractions and, correspondingly, in an inverse order.

It is evident that when increasing the order r of the Farey sequence it is necessary to insert an indefinitely increasing set of ratios of large integers between the ratios of small integers. In this case, only the ratios of small integers correspond to regular STs at the system output [4], and they are to correspond to downward

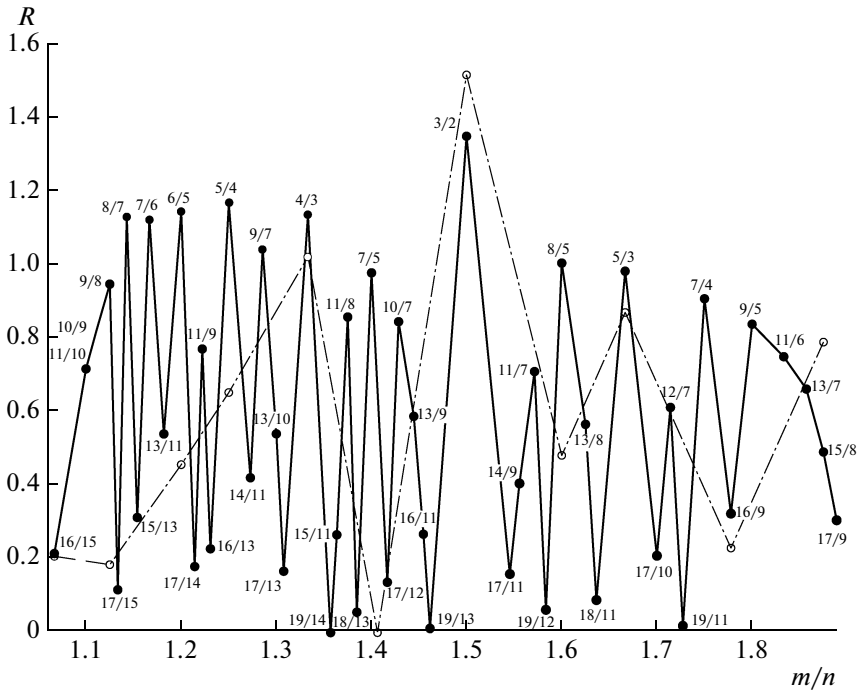

Fig. 2. Regularity of the output spike train as a function of the ratio of input frequencies Ω_1/Ω_2, where $\Omega_2 = 0.4$, $\Omega_1 = (m/n)\Omega_2$. Solid line is a theoretical result. Chain line shows a "rough" estimate obtained using Eq. (2). Black dots are marked by corresponding frequence ratios. The ratios 16/15, 9/8, 6/5, 5/4, 4/3, 3/2, 8/5, 5/3, 16/9, and 15/8 correspond to so-called "just intonation" frequency ratios in music.

excursions on the plotted dependence of the specific entropy 1 on the ratio m/n. Thus, speaking about the regularity, let us introduce it here, analogously to [16], in the form of "inverted" entropy:

$$R(m/n) = \max_{m,n} H(m/n) - H(m/n),$$

where $H(m/n)$ is the specific entropy of an output ST at $\Omega_1/\Omega_2 = m/n$. However, the dependence $R(m/n)$ is not normalized in this case to $\max_{m,n} H(m/n)$ (see [16]), since the entropy is found only for some m/n ratios, and its global maximum, generally speaking, is unknown. We can expect a certain set of regularity peaks; this is demonstrated in Fig. 2 by the solid line with black "dots." The dependence was obtained for $\Omega_2 = 0.4$ (recall that characteristics of the analyzed system are assumed to be dimensionless [4, 6]). Let us discuss some evident features of $R(m/n)$.

(1) As earlier [4], note that the ratio of small integers turns out to be more regular. In this case, we can specifically speak about "small" integers, namely $m, n < 10$.

(2) The ratios of small integers within the given interval are situated more densely at its edges and are more rarefied in the middle. This results in the fact that the most regular consonance "3/2" turns out to be the most isolated one: there are only irregular dissonances in its sufficiently broad vicinity. Such isolation ensures the good distinguishability of the structure of an output ST for "3/2" consonance. Actually, when m/n is slightly tuned out from 3/2 and the observation time is not very long, a deterministic input signal can be hardly distinguished from "3/2" accord; the noise action hides this difference at the output even more strongly. Intensification of the off-tuning yields greater distortion of the features of a deterministic structure of an input signal in an output ST. However, if we imagine that there is some hypothetical ST "structure analyzer," up to a certain value of m/n off-tuning from 3/2 it will signal that the ratio of input oscillations is about 3/2. The situation with the rational numbers situated closer to unison is quite different: if, e.g., we decrease m/n from 6/5, a ST structure will soon start to correspond to the less regular accord "7/6," and then "8/7." Thus, an off-tuning in this range yields a faster qualitative changes in a ST structure.

(3) The accords with small m and n in the frequency ratio yield more regular STs if m/n is to the left of"3/2" in contrast to similar ratios to the right of "3/2." Thus, $R(5/4) > R(5/3)$, $R(8/7) > R(8/5)$, etc. Hence, the regularity of an output system signal depends not only on the smallness of the frequency ratios of an input accord, but also on the m/n location with respect to the unison "1/2," the perfect fifth "3/2," and the octave "2/1."

An analogous dependence $R(m/n)$ can be obtained for other fixed frequencies Ω_2. For example, Fig. 3 plots the regularity dependence on the frequency ratios at $\Omega_2 = 0.5$. It is seen that its behavior corresponds to that described earlier.

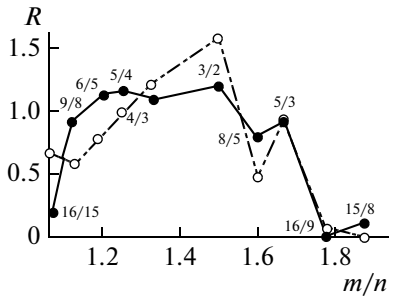

Fig. 3. Regularity of the output spike train as a function of the ratio of input frequencies Ω_1/Ω_2, where $\Omega_2 = 0.5$, $\Omega_1 = (m/n)\Omega_2$. Solid line is a theoretical. Chain line shows a "rough" estimate obtained using Eq. (2). There are no rational ratios with numerators m and denominators n smaller than those shown in the figure within the analyzed range.

4. A COMPARISON WITH NUMERICAL ESTIMATES

To verify the obtained results it is necessary to obtain the dependences shown in Figs. 2 and 3 using a direct numerical estimation of the informational entropy of output STs. However, in this case we face some technical problems related to underestimation of conditional entropy of high orders [1] required for describing a sequence with statistically dependent elementary symbols, namely interspike intervals. Even specialized methods of estimation, e.g., that proposed in [17], turn out to be inefficient. Thus, a simplified approach is proposed, namely a "rough" estimation of entropy under the assumption of independent identically distributed ISIs. In this elementary case, ST entropy is directly found from output probabilistic ISI distribution ρ(t) [4,5] with the equation

$$\tilde{H} = -\int_0^\infty \rho(t)\log_2\rho(t)dt. \qquad (2)$$

The chain line with white "dots" in Fig. 2, 3 shows $\tilde{R}(m/n) = \max_{m/n}\tilde{H}(m/n) - \tilde{H}(m/n)$. It can be easily seen that, on the whole, $\tilde{R}(m/n)$ reflects the dependence $R(m/n)$ of the output ST regularity on the ratio of input frequencies well. Hiding many aforementioned peculiarities in $R(m/n)$ behavior, the $\tilde{R}(m/n)$ estimate nevertheless confirms the result obtained based on the theory of conditional Markov processes.

CONCLUSIONS

A sequence of points of the dependence of an output pulse signal regularity in a neural model on the ratio of frequencies of input sinusoidal actions Ω_1 and Ω_2 at constant intensities of sources of additive white Gaussian noises was obtained with the help of numerical–analytical methods. First, the obtained dependence shows the relative positions of the frequency Ω_1 for which a regular output system signal is observed at the fixed value of Ω_2. Second, it demonstrates obvious quantitative regularity characteristics, i.e., makes it possible to classify consonances more subtly as compared to their rough division into regular (consonance) and irregular (dissonance) types.

In spite of its use of the terms of the theory of music and the theory of sound signal perception by mammals, where the obtained results can find direct application for explaining experimental observations, the present study was carried out within the general theory of random processes and can be useful for studying the mechanisms of signal/noise interactions on the nonlinearities of different systems.

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REFERENCES

1. R. L. Stratonovich, *Information Theory* (Sov. Radio, Moscow, 1975) [in Russian].
2. W. Gerstner and W. M. Kistler, *Spiking Neuron Models. Single Neurons, Populations, Plasticity* (Cambridge Univ., Cambridge, 2002).
3. J. G. Nicholls, A. R. Martin, B. G. Wallace, and P. A. Fuchs, *From Neuron to Brain* (Sinauer, Boston, MA, 2001; Mir, Moscow, 2003).
4. Yu. V. Ushakov, Vestn. NNGU, Radiofiz. **6** (2010, in press).
5. A. N. Burkitt, Biol. Cybern. **95**, 97 (2006).
6. Yu. V. Ushakov, A. A. Dubkov, and B. Spagnolo, Phys. Rev. E **81**, 041911 (2010).
7. Yu. V. Ushakov and A. A. Dubkov, Moscow Univ. Phys. Bull. **65** (5), 372 (2010).
8. R. L. Stratonovich, *Conditional Markov Processes and Their Application to the Theory of Optimal Control* (Elsevier, New York, 1968; Mosk. Gos. Univ., Moscow, 1965).
9. L. R. Rabiner, Proc. IEEE **77**, 257 (1989).
10. A. V. Korolev and A. M. Silaev, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **48**, 358 (2005).
11. P. Cariani, J. New Music Res. **30**, 107 (2001).
12. B. Heffernan and A. Longtin, J. Neurosci. Meth. **183**, 95 (2009).
13. H. E. Plesser and T. Geisel, Phys. Rev. E **63**, 031916 (2001).
14. V. I. Tikhonov and M. A. Mironov, *Markov Processes* (Sov. Radio, Moscow, 1977) [in Russian].
15. *Mathematic Encyclopedy*, Vol. 5, Ed. by I. M. Vinogradov (Moscow, 1985) [in Russian].
16. D. E. Postnov, D. V. Setsinskii, and O. V. Sosnovtseva, Pis'ma Zh. Tekh. Fiz. **27** (11), 49 (2001) [Tech. Phys. Lett. **27**, 463 (2001)].
17. P. Grassberger, arXiv:physics/0307138v2 (2008), http:arxiv.org/abs/physics/0307138v2.