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Methods of Complex Analysis in Model Calculations of the Magnetospheres of Relativistic Stars

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Abstract—In this paper the two-dimensional problem is solved for the magnetosphere of a relativistic compact star. The direct and reverse currents in the magnetotail are calculated, as well as the force exerted by the magnetic field on the edge of the current layer. The dependence of the solution on the parameters of the model is investigated.

Keywords: mathematical methods in physics, conformal mapping, compact star, magnetosphere.

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INTRODUCTION

An important feature of relativistic stars is their strong magnetic field, which exerts great influence on the motion of the plasma near the star. It is responsible for the various observational manifestations of these objects, such as X-rays and mass ejections. Therefore, the study of the magnetosphere of a star (its shape and size, configuration and field size inside it) is one of the important astrophysical problems.

The theory of complex variable functions is successfully applied to solve the two-dimensional problems of cosmic magnetohydrodynamics (see the review in [1]). Below we consider the problem of the magnetosphere of a compact star after the passage of a shock wave caused, for example, by a supernova explosion. Such a shock wave deforms the magnetosphere and drags a part of its magnetic flux. Section 1 presents the mathematical formulation of the problem. Section 2 deals with the method of solving the problem and calculations of the shape of the magnetosphere and magnetic field configuration within it. In Section 3 we calculate the electric currents in the magnetotail and analyze the dependence of the obtained solution on the model input parameters. The magnetic force acting on the edge of the current layer is calculated in Section 4.

1. FORMULATION OF THE PROBLEM

It is assumed that in the two–dimensional approximation the physical picture of the phenomenon may be described by a mathematical model (Fig. 1, on the right). A compact star is replaced by an upward-directed point magnetic dipole with the moment

$$\mathbf{m} = m\mathbf{e}_y. \quad (1)$$

Here $\mathbf{e}_y$ is the unit vector along the vertical axis. The magnetic field is divergence-free and potentially

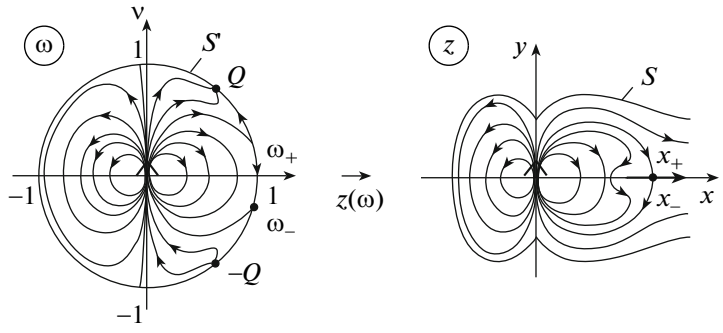

Fig. 1. Magnetic field lines in the auxiliary plane w (on the left) and in the plane z (on the right). S' and S are magnetopauses in the corresponding planes. Q refers to magnetic masses, w_+ and w_- are zero points of the magnetic field in the plane w, which under the mapping $z(w)$ are taken to zero points x_+ and x_-, respectively.

$$\mathrm{div}\mathbf{B} = 0, \quad \mathrm{rot}\mathbf{B} = 0. \tag{2}$$

At the boundary of the magnetosphere S, the internal magnetic pressure is balanced by the external gas pressure

$$B^2/(8\pi)|_S = p. \tag{3}$$

In general, outside the magnetosphere $\mathbf{B} = 0$, equations of the usual gas dynamics may be applied. Assume that the field of a star cannot penetrate the boundary S, i.e.,

$$\mathbf{B} \cdot \mathbf{n}|_S = 0. \tag{4}$$

We believe that the magnetic moment $m = m_0$ and the value of the gas pressure $p = p_0$ are given. Now, we need to find the shape of the boundary S and the magnetic field $\mathbf{B}$ within the magnetosphere.

2. SOLUTION METHOD

A method for solving two-dimensional problems of the Earth's magnetosphere with an a priori unknown boundary was proposed long ago [2–4] and tested in for the magnetospheres of compact stars with an accretion disk, for example, in [1]. The essence of the method is as follows. At any fixed position of the boundary S the solution of equations (2) with conditions (1) and (4) is unique and therefore the condition (3) is sufficient to determine the form of S.

Consider the flat case $\mathbf{B} = (B_x(x, y), B_y(x, y), 0)$. Let $z = x + iy$ be the complex plane. Then it is convenient to describe the field by the potential $F(z)$, an analytic function associated with the vector by the equation [5]

$$\mathbf{B} = B_x + iB_y = -i(dF/dz)^*. \tag{5}$$

The asterisk denotes complex conjugation.

Assume that a conformal mapping exists of the region occupied by a magnetosphere with an unknown boundary S in the plane z, on some known simple area with the given boundary S' in the auxiliary plane $w = u + iv$. This requires that the map $w(z)$ took the origin $z = 0$ into the origin $w = 0$ and kept the slope angle of the dipole. Then, upon constructing the potential $F(w)$, in the plane w created by the dipole $\mathbf{m}$, so that the boundary S' was the field line, and knowing the dependence $z = z(w)$, we may obtain the magnetic field vector in the plane z by the formula (5). Thus, to solve the problem (1)–(4) it is necessary to build the potential $F(w)$ and the map $z(w)$.

Let us use dimensionless variables obtained by dividing the dimensional quantities, i.e., the magnetic moment m, the pressure p, the magnetic field B and distances x and y by m_0, p_0, $B_0 = (8\pi p_0)^{1/2}$, and L_0, respectively. As an auxiliary region in the plane w, we choose the unit circle. Then the required potential will have the following form

$$F(w) = iQ\ln\frac{w - e^{i\alpha}}{we^{i\alpha} - 1} + \left(w - \frac{1}{w}\right). \tag{6}$$

Here Q is the so-called "magnetic mass" [4], it characterizes the magnetic flux carried away by the shock wave in the tail of the magnetosphere. α is another free parameter of the problem, such that the arc $(-\alpha, \alpha)$ of the unit circle in the plane w transfers when mapped into the surface of a neutral current layer in the plane z, whereas the arc $(\alpha, 2\pi - \alpha)$ in the plane w corresponds to the outer boundary of the magnetosphere in the plane z. The configuration of the magnetic field generated by the potential (6) is shown in Fig. 1, on the left.

Condition (3) in dimensionless variables takes the form

$$B^2\big|_S = 1.$$

Hence, using (5), we obtain an ordinary differential equation, which holds at the outer boundary of the magnetosphere

$$\left|\frac{dz}{dw}\right|^2 = \left|\frac{dF}{dw}\right|^2.$$

Since in the plane w the boundary is the unit circle $u^2 + v^2 = 1$, according to the rules of differentiation of the complex function, from the last equality we obtain the following equation for determining the real part of the mapping function on the boundary

$$\left(\frac{dx}{du}\right)^2 = u^2\left|\frac{dF}{dw}\right|^2\Bigg|_{w = u + i\sqrt{1-u^2}}.$$

The equation was solved numerically using the Runge–Kutta method. To calculate the shape of the border, it is necessary to find the imaginary part of the map, the function $y(u)$. We use the fact that the mapping is conformal, i.e., the function $z(w)$ is analytic. Consequently, its real and imaginary parts are harmonically conjugate functions.

By means of the replacement of $u = \cos\varphi$, expansion of $x(\varphi)$ in a Fourier series, and taking the harmonic conjugate series, $y(\varphi)$ is found. Thus, the mapping function for the outer boundary of the magnetosphere is constructed (Fig. 1, on the right).

Let us determine the mapping function $z(w)$ for the segment corresponding to the current layer $(-\alpha, \alpha)$ in the magnetotail. Because of the symmetry of the problem with respect to the x-axis, this layer will be flat. Consequently, $y(\varphi) = 0$ when $\varphi \in (-\alpha, \alpha)$. The arc $(0, \alpha)$ is mapped on the upper side of the layer (i.e., on the upper edge of the relevant section of the complex plane), and the arc $(0, -\alpha)$ is mapped on the lower side of the layer.

Thus, the system of functions $y(\varphi)$ is fully determined for all $\varphi \in (0, 2\pi)$. Now, using the Schwartz formula [6]

$$z(w) = -\frac{1}{2\pi i}\int_0^{2\pi} y(\varphi)\frac{e^{i\varphi} + w}{e^{i\varphi} - w}d\varphi, \tag{7}$$

let us find the function $z(w)$ in the whole circle $|w| \le 1$ (the formula integration was performed numerically). Knowing the potential $F(w)$ and the map $z(w)$, taking

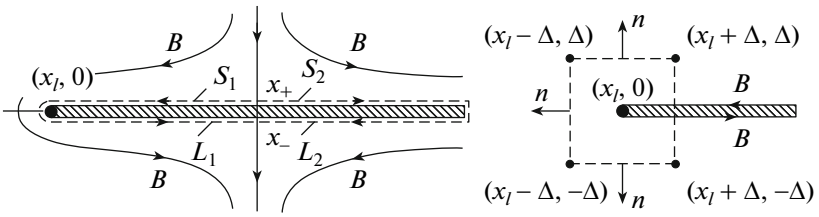

Fig. 2. The configuration of the magnetic field in the vicinity of the current layer. *On the left* the scheme for the calculation of electric currents is shown. x_l is the starting point of the layer, x_+ is the zero point at the upper boundary of the layer, x_- is the zero point at its lower boundary. L_1 is the contour bounding the surface S_1, through which the reverse current $I_{\rm rc}$ is calculated. L_2 is the contour bounding the surface S_2, through which the direct current $I_{\rm dc}$ is calculated. *On the right* there is the scheme for calculating the magnetic force $\mathbf{F}_m$, acting on the edge of the current layer. The dotted line shows the path on which the integration is performed, **n** refers to the external normals to the contour, Δ is the half of the square side.

the unit circle of the plane w into the magnetosphere in the plane z, we obtain the configuration of the magnetic field in the plane z (Fig. 1, on the right).

Let us explain the meaning of the parameter α. For a fixed value of the magnetic flux Q, carried to infinity, α identifies a particular distance from the star before the beginning of the current layer. Indeed, in the plane w the point $w = 1$ corresponds to the beginning of the layer. When mapped, it is taken to the point $z = x_l$, which may be found from the special case of the equation (7). Now, we write it for the points of the unit circle $w = e^{i\varphi}$ and separate the real part

$$x_l = \frac{1}{2\pi}\int\limits_{\alpha}^{2\pi-\alpha} y(\varphi)\cot\frac{\varphi}{2}d\varphi.$$

Here, it is taken into account that the function $y(\varphi)$ vanishes on the segment $(2\pi - \alpha, \alpha)$. For example, note that when choosing $Q = 1/6$ and $\alpha = \pi/3$, the current layer begins at the point $x_l = 0.20$, but with decreasing α the distance from the star to the current layer increases, reaching $x_l = 0.48$ for $\alpha = \pi/6$.

Let us dwell on the question of the location and number of zero points in the magnetic field within the magnetosphere. Such points arise on the current layer as a consequence of the current partition into two branches, i.e., one part of it is carried away to infinity, while the other remains closed on the dipole (Fig. 1, on the right). Reaching the current layer, the total flux spreads along it in opposite directions. Accordingly, at the point of divergence the magnetic field vanishes. A more detailed picture of the magnetic field in the neighborhood of the zero points on the current layer is shown in Fig. 2, on the left. Calculations show that depending on the parameters α and Q, the magnetosphere either contains two zero points on the upper and lower edges of the current layer, or these points coincide with the beginning of the current layer, or they are located between the star and the current layer.

Let us determine for which values of the parameters α and Q the zero point of the field coincides with the inner edge of the current layer. Obviously, zero points in the plane w correspond to zero points in the plane z. Indeed, from the equation (5) follows

$$\mathbf{B} = -i\left(\frac{dF}{dz}\right)^* = -i\left(\frac{dF}{dw}\frac{dw}{dz}\right)^* = 0.$$

The derivative dw/dz is nonzero due to conformality. Therefore, the following equality should hold true

$$dF/dw = 0, \qquad (8)$$

implying that components B_u and B_v of the magnetic field in the plane w are zero. The potential F in this plane has a convenient form for the analysis (6). Because of the symmetry of the magnetic field relative to the x-axis in the auxiliary plane w the point (1,0) corresponds to the beginning of the current layer. Substituting these coordinates into equation (8), after separating the real and imaginary parts, we obtain a system of two algebraic equations with respect to two variables α and Q. The solution of this system shows that the zero point of the field coincides with the inner edge of the current layer, if the parameters are chosen so that

$$Q_{\max} = 2\tan\frac{\alpha}{2}. \qquad (9)$$

Once again we see that $Q = 0$ corresponds to $\alpha = 0$, and at small $Q \neq 0$ we have $Q \approx \alpha$.

The method makes it possible to obtain estimates of the magnetic field intensity at any point of the magnetosphere.

To return to dimensional variables, it is necessary to set the values of the magnetic moment m_0 and pressure p_0 of the interstellar gas at the boundary of the magnetosphere. For a typical neutron star in the interstellar gas it takes the following form

$$m_0 = 10^{30}\ \text{G/cm}^3, \quad p_0 = 1.38\times10^6\ \text{dyne/cm}^3. \qquad (10)$$

3. ELECTRIC CURRENTS IN THE MAGNETOTAIL

The circulation of the magnetic field relative to the current layer determines the amount of electric current in it. In the case where the zero points lie on the current layer, there are two oppositely directed currents, i.e., one is determined by the circulation around the inner part of the layer to zero points x_+ and x_-, and

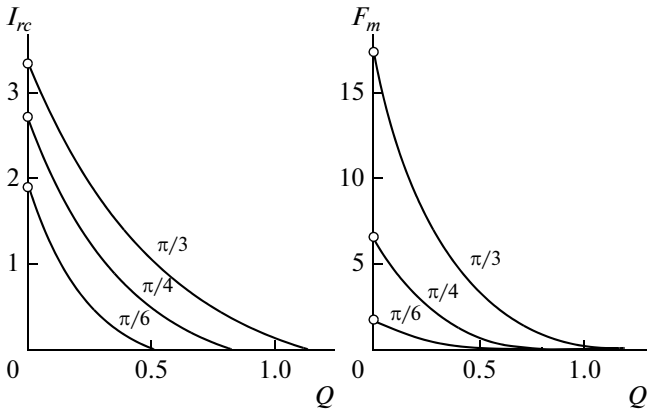

Fig. 3. The dependence of the dimensionless functions I_{rc} (on the left) and F_m (on the right) on the parameter Q for three different values of α. The values of α are given next to the curves.

another is determined by the circulation around the outside part of the layer from zero points x_+ and x_- to infinity (Fig. 2, on the left). The first one is called the reverse current, the second is called the direct current.

In dimensionless variables, the current density is

$$\mathbf{j} = \xi^{-1}\mathrm{rot}\mathbf{B}, \text{ where } \xi = 4\pi j_0 L_0/cB_0.$$

Here c is the speed of light, and j_0 is the typical current density. Let $j_0 = cB_0/4\pi L_0$. Then we obtain $\xi = 1$, which will simplify the calculations.

First, we calculate the reverse current, i.e., the current through the area S_1, covering the inner edge of the current layer to zero points x_+ and x_- (shown by dotted line in Fig. 2, on the left).

$$I_{rc} = \int_{S_1}\mathbf{j}d\mathbf{s} = \int_{S_1}\mathrm{rot}\mathbf{B}d\mathbf{s} = \oint_{L_1}\mathbf{B}d\mathbf{l},$$

where L_1 is the contour bounding the area S_1. Let us choose L_1 such that it is as close to the current layer as possible. Then, using formula (5) we obtain

$$I_{rc} = \mathrm{Im}\{F(z)\}\big|_{z=x_+} - \mathrm{Im}\{F(z)\}\big|_{z=x_-}.$$

Since the mapping $z(w)$ takes zero points in the plane w to zero points of the plane z, then from the last equality we have

$$I_{rc} = \mathrm{Im}\{F(z[w])\}\big|_{w=w_+} - \mathrm{Im}\{F(z[w])\}\big|_{w=w_-}$$
$$= \mathrm{Im}\{F[w_+]\} - \mathrm{Im}\{F[w_-]\}.$$

Note that the points of the unit circle $w_\pm = \exp(i\varphi_\pm)$. Since $\varphi_+ = -\varphi_-$ due to the symmetry of the problem relative to the axis of x, we denote $\varphi_+ = -\varphi_- \equiv \varphi_0$ and rewrite the last formula as

$$I_{rc} = \mathrm{Im}\{F[\exp(i\varphi_0)]\} - \mathrm{Im}\{F[\exp(-i\varphi_0)]\}$$
$$= 4\sin\varphi_0 + \frac{1}{2}Q\ln\frac{\cos(\alpha-\varphi_0)-1}{\cos(\alpha+\varphi_0)-1}. \quad (11)$$

For a given α and Q the angle φ_0 is uniquely determined by the condition $\mathbf{B} = 0$ using formulas (5) and (6). Thus, the formula (11) allows us to calculate the value of the reverse current for all values of the parameters α and Q. The magnitude of the reverse current is maximal for any finite values of Q.

Similarly, we obtain the expression for the direct current

$$I_{dc} = [\mathrm{Im}\{F[\exp(i\alpha)]\} - \mathrm{Im}\{F[\exp(i\varphi_0)]\}]$$
$$+ [\mathrm{Im}\{F[\exp(-i\varphi_0)]\} - \mathrm{Im}\{F[\exp(-i\alpha)]\}] \quad (12)$$
$$= -I_{rc} + [\mathrm{Im}\{F[\exp(i\alpha)]\} - \mathrm{Im}\{F[\exp(-i\alpha)]\}].$$

Points α and $-\alpha$ are singular points of the potential (6), magnetic masses are situated in them. It is easy to verify that the second term is the infinite value. Thus, the direct current at a given model formulation of the problem is infinite.

Consider the dependence of the reverse current on the model parameters. Let us return to dimensional variables

$$I_{rc}^{\mathrm{dim}} = \frac{c}{4\pi}B_0 L_0 I_{rc}(\alpha, Q). \quad (13)$$

Hence, it is clear that the magnitude of the reverse current depends on four parameters: (1) the gas pressure at the boundary of the magnetosphere p_0, (2) the magnetic moment of the star m_0, (3) the magnetic flux Q, and (4) the angle α. The function $I_{rc}(\alpha, Q)$ is shown in Fig. 3 on the left, demonstrating that for a fixed Q the greater α, the greater the current, and for a fixed α the smaller Q, the greater the current.

Note that when $Q = 0$, the solution is discontinued, i.e., the reverse current becomes zero, since the current layer ceases to exist. This fact is reflected in the formula for the potential (6), where the first term, which is responsible for the tail of the magnetosphere, disappears and only the part related to the dipole remains. With such a potential the closed magnetosphere without a current layer satisfies the boundary condition (3) [7].

Note also that for any fixed α the reverse current is nonzero only if values of Q lie in a finite interval $(0, Q_{\max})$. Indeed, the increase in Q means the increase in the flux, carried away to infinity in the tail of the magnetosphere. Accordingly, the part of the flux that is closed on the dipole decreases and the points of divergence of fluxes (zero points x_- and x_+) move closer to the dipole, i.e., to the inner edge of the current layer. At the time when they reach the edge, the reverse current vanishes. Therefore, condition (9) may be viewed as the vanishing condition of the reverse current; at $Q > Q_{\max}$ the reverse current does not exist.

The dependence of the reverse current value on the moment of the star and the gas pressure is natural; the greater these values are, the greater the current is. Since after the passage of the shock wave, the pressure behind its front increases sharply, then choosing the value of p_0 the same as before (see (10)), we obtain the

lower limit of the reverse current. For example, if $\alpha = \pi/3$ and $Q = 1/6$, this value is $I_{rc}^{dim} = 6 \times 10^{12}$ Amps. As the pressure behind the front of the shock wave of large amplitude may be several orders of magnitude greater than the initial pressure, the reverse current may be much larger than the specified minimum value, which is important for the astrophysical application of the obtained solution.

4. MAGNETIC FORCE ACTING ON THE EDGE OF THE CURRENT LAYER

Let us calculate the force exerted by the magnetic field on the inner edge of the current layer. Such a force per unit volume in dimensionless variables has the following form

$$\mathbf{f}_m = \mathbf{j} \times \mathbf{B} = \mathrm{rot}\mathbf{B} \times \mathbf{B} = (\mathbf{B} \cdot \mathrm{grad})\mathbf{B} - \mathrm{grad}\, B^2/2.$$

To find the resultant force acting on the final volume, we may integrate f_m over the entire volume, or we may reduce the integral to the volume to a surface integral:

$$\mathbf{F}_m = \int_V \mathbf{f}_m dV = \int_S \mathbf{B} \cdot (\mathbf{B} \cdot ds) - \int_S B^2/2 d\mathbf{s}. \qquad (14)$$

Here S is a surface that limits the volume of V and $d\mathbf{s}$ is a vector equal in magnitude to the surface of the element and perpendicular to it. Since we consider the two-dimensional case, we take the contour surrounding the inner edge of the current layer as the surface. To simplify the calculations, we choose a square with a side of 2Δ and with the center at the beginning of the layer at the point x_l as the contour (see Fig. 2, on the right). Further, the formula (5) may be represented as

$$\mathbf{B} = -i\left(\frac{dF}{dw}\frac{dw}{dz}\right)^*. \qquad (15)$$

Here dF/dw is a known analytical function. To find the function dw/dz in a small neighborhood of the current layer it is possible to avoid numerical integration of the Schwartz formula. Indeed, the map $z(w)$, taking the arc of the unit circle $(-\alpha, \alpha)$ in the plane w to the edge $(x_l, +\infty)$ in the plane z (Fig. 4) has the form

$$z = \frac{x_l}{z_2} = x_l\left(\frac{z_1^2 + 1}{2z_1}\right)^{-1} = x_l\frac{2z_1}{z_1^2 + 1} = x_l\frac{2w^{\pi/2\alpha}}{w^{\pi/\alpha} + 1}.$$

Hence, we find

$$w = \left(\frac{x_l \pm \sqrt{x_l^2 - z^2}}{z}\right)^{2\alpha/\pi}. \qquad (16)$$

Substituting the mapping of (16) into (15), we obtain an analytic expression for the magnetic field components in the plane z, which simplifies the calculation of the integral (14). Calculations show that the magnetic force is directed along the axis of x, $\mathbf{F}_m = F_m\mathbf{e}_x$. Figure 3 presents graphs of the behavior of F_m,

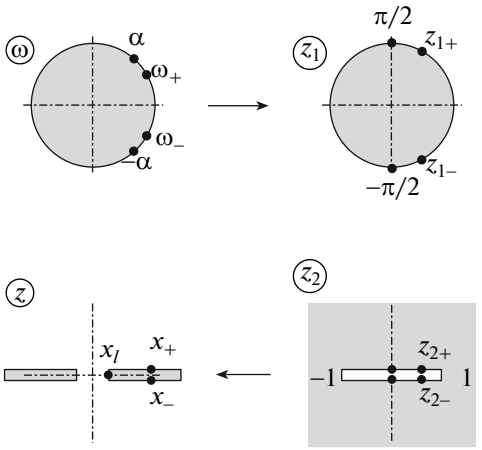

Fig. 4. The scheme for the construction of the mapping that takes the arc of the unit circle $(\alpha, -\alpha, \alpha)$ in the auxiliary plane w to the edge $(x_l, +\infty)$ in the plane z.

depending on the parameter Q for three different values of α. With increasing Q the magnetic force will decrease and vanish at $Q = Q_{max}$, when the zero point reaches the edge of the disc (see Section 3). As before, when $Q = 0$ the current layer disappears and, consequently, the power F_m vanishes. Note that in the case of previously selected parameters $\alpha = \pi/3$ and $Q = 1/6$ we obtain the dimensionless value of the force $\mathbf{F}_m = 10.8\mathbf{e}_x$. This is the lower limit of the magnetic force, since the pressure increases by many times behind the front of the shock wave.

CONCLUSIONS

In a two-dimensional approximation, the configuration is calculated of the magnetic field inside the magnetosphere of a relativistic star after the passage of a shock wave caused, for example, by a supernova explosion. Such a shock wave deforms the magnetosphere, dragging part of its magnetic flux. As a result, the current layer occurs in the magnetotail.

Direct (I_{dc}) and inverse (I_{rc}) electrical currents that occur in this layer are calculated, as well as the power F_m, exerted by the magnetic field on the inner edge of the current layer. The dependencies of these quantities on the parameters involved in the problem are presented in graphical form. It is shown that

(1) In our formulation of the problem, the direct current is infinite, $I_{dc} = \infty$.

(2) At a fixed distance from the star to the inner edge of the current layer I_{dc} and F_m are limited and decrease with the increase of the magnetic flux carried away to infinity. An analytical formula is obtained for calculating the maximum value of the magnetic flux, at which the reverse current and the magnetic force vanish.

(3) For a fixed current, carried to infinity, I_{dc} and F_m decrease with increasing distance between the star and the edge of the current layer.

The dimensionless quantities provided in the graphs may be used to determine the appropriate dimensional quantities for arbitrary compact objects if the magnetic moment of the star m_0 and the gas pressure at the boundary of the magnetosphere p_0 are given. The lower limits of the dimensional quantities of the reverse current and magnetic force in the magnetosphere of a standard neutron star are listed.

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