ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 6, pp. 547–549. © Allerton Press, Inc., 2010.
Original Russian Text © Yu.S. Vernov, M.N. Mnatsakanova, S.G. Salinsky, 2010, published in Vestnik Moskovskogo Universiteta. Fizika, 2010, No. 6, pp. 113–115.

A New Definition of Regularity for Representation of Canonical Commutation Relations Algebras

Yu. S. Vernov[a, b], M. N. Mnatsakanova[b], and S. G. Salinsky[c]

[a] *Institute for Nuclear Research, Russian Academy of Sciences, V-312, Prospect Shestidesyatiletiya Oktyabrya, 7a. Moscow, 117312 Russia.*
[b] *Institute of Nuclear Physics, Moscow State University, GSP-1, Moscow, 119991 Russia.*
[c] *Department of Quantum Theory and High Energy Physics, Faculty of Physics, Moscow State University, Moscow, 19991 Russia.*
e-mail: vernov@inc.ac.ru, mnatsak@theory.sinp.msu.ru

Received July 4, 2010

Abstract—A new criterion of regularity for the representation of canonical commutation relations algebras is given on the basis of the concept of an analytical vector.

Keywords: Quantum theory, canonical commutation relations.
DOI: 10.3103/S0027134910060226

INTRODUCTION

The basis of quantum mechanics is the algebra of canonical commutation relations (CCR) and a major role is played by its representations, which are called regular (different definitions of regular representation of CCR will be given below). An overview of CCR was made in [1]. In this paper we give a new regularity criterion for representation of the CCR based on the notion of an analytic vector, as introduced in [2].

In the simplest case of one-dimensional CCR we have the form:

$$[\hat{p}, \hat{q}] = -i\hat{I}, \qquad (1)$$

where $\hat{p}$ and $\hat{q}$ are self-adjoint operators (in quantum mechanics the momentum and coordinate operators, respectively).

Note the well-known fact that in the case of an arbitrary, but finite, number of operators, i.e., when

$$[\hat{p}_i, \hat{q}_k] = -i\delta_{ik}; \quad 1 \le i, k \le n,$$

we have the same pattern as in the case of two operators $\hat{p}$ and $\hat{q}$. Taking this into account, consider the case where the equality (1) holds. We emphasize that in the case of an infinite number of operators, i.e., in the case of quantum field theory, we have a more complex pattern, the consideration of which is beyond the scope of this paper.

The most well-known representation of CCR is the Schrodinger representation, which is realized in the space $L_2(-\infty, +\infty)$, i.e., in the space of functions $f(x)$, such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. This representation is regular. In it the operators $\hat{p}$ and $\hat{q}$ are defined as follows:

$$\hat{q}f(q) = qf(q), \quad \hat{p}f(q) = -i\frac{\partial}{\partial q}f(q). \qquad (2)$$

One definition of the regularity of the representations of CCR algebra, which uses the regularity of the Schrodinger representation, is the following:

Definition 1. Any representation of CCR algebra that is unitarily equivalent to the Schrodinger representation, is regular.

Note one important fact: there is no realization of the representation of CCR algebra by bounded operators [1]; at least one of the operators $\hat{p}$ and $\hat{q}$ should be unlimited. Recall that in a closed space unbounded operators cannot be defined in all space, but only in a dense domain.

There are different definitions of the regularity of representation of CCR algebra. We emphasize that we are talking about representations in Hilbert space. However, CCR is also considered in a space with an indefinite metric [3]. Note that in the covariant gauge field theory the inevitable transition from the Hilbert space to a space with an indefinite metric occurs [4–5].

An important contribution to the description of regular representations of the CCR algebra was made by the Rellich–Dixmier theorem, which shows that the representations of CCR algebra are regular for a very large class of operators (see [1]).

Rellich–Dixmier theorem. The self-adjoint operators $\hat{p}$ and $\hat{q}$ form a regular representation of CCR algebra, if:

1. There exists a dense domain $D = D(\hat{p}) \cap D(\hat{q})$ in which the relation (1) holds.

2. Operator $(\hat{p}^2 + \hat{q}^2)$ is a self-adjoint operator in D.

Note that Fuglede constructed an example of a non-regular representation in the case where only condition 1 holds [6].

A simple example of an irregular representation of the CCR algebra is the representation given by the operators (2), but in the space $L_2(a, b)$.

CCR can also be specified in the form:

$$[\hat{a}, \hat{a}^*] = \hat{I}, \tag{3}$$

where the operator $\hat{a}$ and its adjoint operator $\hat{a}^*$ are equal, respectively, to

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \text{ and } \hat{a}^* = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}). \tag{4}$$

It is easy to verify that in the regular representation the operator $\hat{N} = \hat{a}^*\hat{a}$ a has $\mathrm{Sp}\hat{N} = \mathbb{N}$ in Hilbert space.

One peculiarity of the regular representations of CCR algebra in Hilbert space is the presence of vacuum vector. Such representations are called Fock representation. Indeed, according to (1) and (4), the vector $\psi_0 = Ce^{-\frac{q^2}{2}}$ satisfies the condition $\hat{a}\psi_0 = 0$ and, consequently, $\hat{N}\psi_0 = 0$. Although all the definitions of regular representations are equivalent, some of them make it easier to study the CCR in more general spaces than Hilbert space, namely, in spaces with indefinite metrics, in particular, in the Krein space [7–8]. For example, the requirement of the existence of eigenvectors of the operator $\hat{N}$

$$\hat{N}\psi_\alpha = \alpha\psi_\alpha \tag{5}$$

leads to a description of regular representations in the Krein space [3].

The fact that the operators $\hat{p}$ and $\hat{q}$ and, respectively, $\hat{a}$ and $\hat{a}^*$ are unbounded operators, leads to technical difficulties associated with identifying domains in which they can be given. This difficulty is eliminated by setting CCR in the Weyl form:

$$e^{it\hat{p}}e^{is\hat{q}} = e^{ist}e^{is\hat{q}}e^{it\hat{p}}. \tag{6}$$

It is known that, according to Stone's theorem, the operators $e^{it\hat{p}}$ and $e^{is\hat{q}}$ are bounded, since $\hat{p}$ and $\hat{q}$ are self-adjoint operators [9].

The Weyl form of CCR is widely used in quantum mechanics (see, e.g., [10]) The representations of CCR in the Weyl form considered in a Hilbert space, but, like representations of CCR in a standard form, it is natural to study the status of the Weyl representation in a space with an indefinite metric, which will be done in our next work.

For this purpose, and perhaps not only for it, a new criterion for the regularity of representations, which is the subject of this paper, may be useful.

ANALYTICAL VECTORS AND THEIR RELATION TO THE REPRESENTATION OF CCR IN THE WEYL FORM

Recall the definition of analytic vectors introduced in [2].

Definition 2. Let $\hat{A}$ be a linear operator in Hilbert space H. The vector $\xi \in H$ is called an analytic vector for the operator $\hat{A}$ if ξ belongs to the defined domain of the operators $\hat{A}^k$, $\forall k \in \mathbb{N}$ and for some $t > 0$ the series converges:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\left\|\hat{A}^k\xi\right\|. \tag{7}$$

In this case we can define the operator $\exp(t\hat{A})$ by its Taylor series

$$e^{t\hat{A}}\xi = \sum_{k=0}^{\infty} \frac{t^k}{k!}\hat{A}^k\xi \tag{8}$$

for those ξ for which the series (8) converges. We will formulate a criterion of the regularity of representation of CCR algebra in the following theorem. We will prove that the representation of CCR algebra is regular if there exists a dense domain D such that any vector $\xi \in D$ satisfies the conditions

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\left\|\hat{q}^k\xi\right\| < \infty, \quad \forall t > 0;$$
$$\sum_{k=0}^{\infty} \frac{s^k}{k!}\left\|\hat{p}^k\xi\right\| < \infty, \quad \forall s > 0 \tag{9}$$

and any regular representation satisfies (9).

We first prove the regularity of the considered representation. The presented proof is similar to the corresponding proof in [1], but without the assumption of boundedness of the operators $\hat{p}$ and $\hat{q}$. From relations (1) it immediately follows that

$$\hat{p}\hat{q}^n - \hat{q}^n\hat{p} = -i(\hat{q}^n)', \quad (' = d/dq). \tag{10}$$

Hence, by (10) and (9)

$$\hat{p}e^{is\hat{q}} - e^{it\hat{p}}\hat{p} = -i(e^{it\hat{q}})'. \tag{11}$$

From (11) it follows immediately that

$$e^{-it\hat{q}}\hat{p}e^{it\hat{q}} = (\hat{p} + t\hat{I})$$

and, consequently,

$$e^{-it\hat{q}}\hat{p}^n e^{it\hat{q}} = (\hat{p} + t\hat{I})^n. \tag{12}$$

Using condition (9) again, we find that

$$U_t V_s = e^{its}V_s U_t, \quad U_t \equiv e^{it\hat{p}}, \quad V_s \equiv e^{is\hat{q}}. \tag{13}$$

Thus, the existence of the Weyl relation (13) in domain D is proved.

The next step is to apply the Weyl representation to the entire space H. It suffices to note that D is a dense

area, and U_t and V_s, according to Stone's theorem, are bounded operators. Note that the Stone theorem imposes several very weak conditions on the groups U_t and V_s ([11]).

It remains to prove that from the implementation of (13) in H follows the pattern of the representation considered. But this result is the content of the von Neumann theorem (see, e.g., [1]). Note that since every regular representation is a direct sum of irreducible representations, it suffices to consider only the irreducible representation.

It remains to prove that every regular representation contains analytic vectors in a dense domain and therefore satisfies the Weyl relation (13) in H. For this it is convenient to use the CCR relation in form (3). In this case,

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^*), \quad \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^*). \tag{14}$$

We will prove that the operator $\hat{q}$ has an analytic vector in the dense domain D. In the regular representation we can construct an orthogonal basis of eigenvectors ψ_n and operator $\hat{N} = \hat{a}^*\hat{a}$. The relations (3) and (5) show that $\mathrm{Sp}\,\hat{N} = \mathbb{N}$. It is easy to see that

$$(\boldsymbol{\psi}_n, \boldsymbol{\psi}_n) = n!, \text{ where } \psi_n = (\hat{a}^*)^n\boldsymbol{\psi}_0, \quad \boldsymbol{\psi}_0 \text{ is the vacuum vector, } (\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) = 1. \tag{15}$$

Further, it is easy to show that

$$(\hat{a}^*\boldsymbol{\psi}_n, \hat{a}^*\boldsymbol{\psi}_n) = (n+1)!, \quad (\hat{a}\boldsymbol{\psi}_n, \hat{a}\boldsymbol{\psi}_n) = n!. \tag{16}$$

The formulas (15) and (16) are well known from the theory of quantum harmonic oscillators, for example, see [12]. Consider the domain D consisting of all the finite linear combinations of the vectors $\boldsymbol{\psi}_n$. Because H consists of all the finite or convergent linear combination of vectors $\boldsymbol{\psi}_n$, then D is a dense domain. The norm of the vector $\boldsymbol{\psi}_n$ can be defined by formula $\|\boldsymbol{\psi}_n\| = \sqrt{(\boldsymbol{\psi}_n, \boldsymbol{\psi}_n)}$. According to (15)

$$\|\boldsymbol{\psi}_n\| = \sqrt{n!}. \tag{17}$$

Accordingly, for the vector

$$\boldsymbol{\psi} = \sum_m^{m+n} C_k\boldsymbol{\psi}_k \tag{18}$$

by (15)–(17), we have the following estimate

$$\|\hat{q}\boldsymbol{\psi}\| \le \sum_m^{m+n} \|C_k\hat{q}\boldsymbol{\psi}_k\| \le \sqrt{2}C\sqrt{(m+n+1)!}, \quad C = \max|C_k|, \quad m \le k \le m+n. \tag{19}$$

For the derivation of (19) we have taken into account that

$$(\hat{a}^*\boldsymbol{\psi}_n, \hat{a}^*\boldsymbol{\psi}_n) = (\boldsymbol{\psi}_n(n+1)\boldsymbol{\psi}_n) = (n+1)(\boldsymbol{\psi}_n, \boldsymbol{\psi}_n),$$
$$(\hat{a}\boldsymbol{\psi}_n, \hat{a}\boldsymbol{\psi}_n) = (\boldsymbol{\psi}_n, n\boldsymbol{\psi}_n) = n(\boldsymbol{\psi}_n, \boldsymbol{\psi}_n).$$

According to (19)

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\|\hat{q}^k\psi\| \le \sum_{k=0}^{\infty} \frac{t^k}{k!}C^k(2(m+n+1)!)^{k/2}. \tag{20}$$

Series (20) obviously converges for any finite m and n.

Thus, it is proved that the vector ψ, $\forall\psi \in D$ is an analytic vector for the operator $\hat{q}$. The proof that any vector $\psi \in D$ is an analytic vector for the operator $\hat{p}$ is similar.

CONCLUSIONS

In this paper a new criterion for the regularity of the representation of CCR algebra based on the notion of analytic vectors is obtained. There is a reason to believe that it will be useful for the study of representations of CCR in the Weyl form in spaces with indefinite metric.

REFERENCES

1. C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics* (Springer, Berlin, Heidelberg, New York, 1967), Ch. 4, p. 63.
2. E. Nelson, Ann. Math. **70**, 572 (1959).
3. M. Mnatsakanova, G. Morchio, F. Strocchi, and Yu. Vernov, J. Math. Phys. **39**, 2969 (1998).
4. G. Morchio and F. Strocchi, Ann. Inst. Poincaré A **33**, 251 (1980).
5. T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. **66**, 1 (1979).
6. B. Fuglede, Math. Scand. **20**, 70 (1967).
7. J. Bognar, *Indefinite Inner Product Spaces* (Springer, Berlin, Heidelberg, New York, 1974).
8. T. Ya. Azizov and I. S. Iokhvidov, *Liner Operators in Space with an Indefinite Metric* (Nauka, Moscow, 1986; Wiley, New York, 1989).
9. K. Yosida, *Functional Analysis* (Springer, Berlin, 1965; Mir, Moscow, 1967).
10. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics,* Vol. 2 (Springer, Berlin, Heidelberg, New York, 1979).
11. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Ungar, New York, 1955; Mir, Moscow, 1979).
12. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics,* Vol. 2: *The Classical Theory of Fields* (Nauka, Moscow, 1988; Pergamon, Oxford, 1975).