OSCILLATIONS OF NONLINEAR SYSTEMS AFFECTED BY SMALL RANDOM PERTURBATIONS

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We consider a mechanical system having a single degree of freedom, whose Hamiltonian, which depends on the slowly varying parameters $x=(x_1,\ldots,x_n)$, is

$$H=\frac{p^2}{2m(x)}+V(x,q).$$

We assume that the system is subjected to small determined and random perturbations such that the behavior of the perturbed motion is described by a system of stochastic Ito equations of the form

$$\begin{gathered} dq=\left[\frac{p}{m(x)}-\varepsilon f^{(p)}(x,q,p)\right]dt+\sqrt{\varepsilon}\,[\sigma_{11}(x,q,p)\,dW_1+\sigma_{12}(x,q,p)\,dW_2], \\ dp=[-Q(x,q)+\varepsilon f^{(q)}(x,q,p)]\,dt+\sqrt{\varepsilon}\,[\sigma_{21}(x,q,p)\,dW_1+\sigma_{22}(x,q,p)\,dW_2], \\ dx=\varepsilon X(x,q,p)\,dt. \end{gathered} \tag{1}$$

Here $Q(x,q)=\dfrac{\partial V}{\partial q}$, $X(x,q,p)$ is an n-dimensional vector function, $W_1(t)$ and $W_2(t)$ are independent Wiener processes, $MW_i=0$, $MW_i^2=t$, ε is a small parameter, and $\varepsilon\in[0,\varepsilon_0]$.

We assume that the unperturbed motion of the system is purely periodic, with a curve T_0 which depends on the parameters x and the initial conditions; the coordinate q has one maximum F_1 and one maximum F_2 during a period. (We could also consider the case in which the perturbed motion is rotational.) Under these assumptions, we can describe the unperturbed motion by $q=q(x,F_1,\psi)$, $p=p(x,F_1,\psi)$, where q and p are periodic functions (with a period of 2π) of the phase $\psi=\dfrac{2\pi}{T_0}(t-t_0)+h$, where h is an arbitrary constant, F_1 is a second arbitrary constant, which we assume below to be one possible maximum deviation of the system from its equilibrium position, and x = const.

We evidently have

$$V(x,F_1)\equiv V(x,F_2)=\frac{p^2}{2m(x)}+V(x,q). \tag{2}$$

If p and q are random processes which satisfy system (1), Eq. (2) gives the two possible (random) maximum deviations F_1 and F_2 from the equilibrium position for any time t.

Below we derive on the basis of an averaging principle [1, 2] average stochastic equations for F_1 and F_2 for the case of perturbed motion of the system. Analogous topics were analyzed in [3, 4].

In system (1) we replace the variables x, q, p by x, F_1, ψ. In this new system of "slow" variables we have F_1 and x. Using the equal equation (/5/, p. 506) and Eq. (2), we find that stochastic equations

$$\begin{gathered} dF_1=\frac{\varepsilon}{Q(x,F_1)}A(x,q,p)\Big|_{\substack{q=q(x,F_1,\psi)\\ p=p(x,F_1,\psi)}}dt+ \\ +\frac{\sqrt{\varepsilon}}{Q(x,F_1)}[B_1(x,q,p)\,dW_1+B_2(x,q,p)\,dW_2]\Big|_{\substack{q=q(x,F_1,\psi)\\ p=p(x,F_1,\psi)}}, \\ dx=\varepsilon X(x,q,p)\Big|_{\substack{q=q(x,F_1,\psi)\\ p=p(x,F_1,\psi)}}dt, \end{gathered} \tag{3}$$

where

$$\begin{gathered} A(x,q,p)=f^{(q)}\frac{p}{m}-f^{(p)}Q-\frac{1}{m}X\frac{\partial}{\partial x}\int_q^{F_1}[m(x)Q(x,y)]\,dy+ \\ +\frac{1}{2}\left[\frac{1}{m}\left(1-\frac{2}{Q^2(x,F_1)}\int_q^{F_1}Q(x,y)\,dy\right)a_{11}-2\frac{Q(x,q)}{Q(x,F_1)}\frac{\partial Q(x,F_1)}{\partial F_1}a_{12}+\right. \\ \left.+\left(\frac{\partial Q(x,q)}{\partial q}-\frac{Q^2(x,q)}{Q^2(x,F_1)}\frac{\partial Q(x,F_1)}{\partial F_1}\right)a_{22}\right],\quad a_{ik}=\sigma_{i1}\sigma_{k1}+\sigma_{i2}\sigma_{k2}, \\ B_1=Q(x,q)\,\sigma_{11}+\frac{p}{m}\sigma_{21},\quad B_2=Q(x,q)\,\sigma_{12}+\frac{p}{m}\sigma_{22}. \end{gathered} \tag{4}$$

According to [1], we can carry out an averaging in group (3). We denote the average values F_1, F_2 and x by $\bar F_1, \bar F_2$, and $\bar x$, :

$$d\bar F_1 = \frac{\varepsilon}{Q(x, \bar F_1)}\left[\frac{1}{T_0}\int\limits_0^{T_0} A(\bar x, q, p)\Big|_{\substack{q=q(\bar x, \bar F_1, \psi)\\ p=p(\bar x, \bar F_1, \psi)}} dt\right] dt +$$

$$+ \frac{\sqrt{\varepsilon}}{Q(\bar x, \bar F_1)}\left[\frac{1}{T_0}\int\limits_0^{T_0} (B_1^2 + B_2^2)\Big|_{\substack{q=q(\bar x, \bar F_1, \psi)\\ p=p(\bar x, \bar F_1, \psi)}} dt\right]^{1/2} dW,$$

$$d\bar x = 2\left[\frac{1}{T_0}\int\limits_0^{T_0} X(\bar x, q, p)\Big|_{\substack{q=q(\bar x, \bar F_1, \psi)\\ p=p(\bar x, \bar F_1, \psi)}} dt\right] dt. \tag{5}$$

We note that the time averaging can be replaced by an integration over q over the segment of the unperturbed trajectory corresponding to the interval $\Delta t = T_0$. (The unperturbed-motion trajectory need not be known). Here we must take that we have

$$\dot q = \pm m^{-1/2}\left[2\int\limits_q^{F_1} Q(x, y)\, dy\right]^{1/2},$$

where the plus sign is used for the case in which q varies from F_2 to F_1, and the minus sign as q varies from F_1 to F_2. The average value of $\bar A(\bar x, \bar F_1)$ in (5) (the average value of the stochastic part of the equation is found in similar manner) is

$$\bar A(\bar x, \bar F_1) = \left(2m^{1/2}\int\limits_{\bar F_2}^{\bar F_1} dq\left(2\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2}\right)^{-1} \times$$

$$\times \int\limits_{\bar F_2}^{\bar F_1}\left\{\sum_{i=1}^{2}(-1)^i f^{(q)}\left(\bar x, q, (-1)^i\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right) - \right.$$

$$- m^{1/2}\left(2\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2} Q\left(\bar x, q\right)\sum_{i=1}^{2} f^{(p)}\left(\bar x, q, (-1)^i\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right) -$$

$$-\int\limits_q^{\bar F_1}\frac{\partial}{\partial x}[mQ(\bar x, y)]\, dy\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2}\sum_{i=1}^{2} X\left(\bar x, q, (-1)^i\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right) +$$

$$+ \int\limits_q^{\bar F}\left(1 - \frac{2}{Q^2(F_1, \bar x)}\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2} \times$$

$$\times \sum_{i=1}^{2}(-1)^i a_{11}\left(\bar x, q, (-1)^i\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right) - \frac{Q(\bar x, q)}{Q^2(\bar x, F_1)} \times$$

$$\times \frac{\partial Q(\bar x, \bar F_1)}{\partial \bar F_1} m^{1/2}\left(2\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2}\sum_{i=1}^{2}(-1)^i a_{1,2}\left(\bar x, q, (-1)^i \times\right.$$

$$\left.\times\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right) + \frac{1}{2}\left(\frac{\partial Q(\bar x, q)}{\partial q} - \frac{Q^2(\bar x, q)}{Q^2(\bar x, F_1)}\frac{\partial Q(\bar x, \bar F_1)}{\partial F_1}\right) m^{1/2} \times$$

$$\left.\times\left(2\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{-1/2}\sum_{i=1}^{2}(-1)^i a_{22}\left(\bar x, q, (-1)^i\left(2m\int\limits_q^{\bar F_1} Q(\bar x, y)\, dy\right)^{1/2}\right\} dq\left(\int\limits_{\bar F_2}^{\bar F_1} Q(\bar x, y)\, dy\right) = 0. \tag{6}$$

It follows from the averaging principle that the probability distribution of F_1 (F_2) for sufficiently small ε will come arbitrarily close to the probability distribution for the quantity $\bar F_1(\bar F_2)$ over a time integral on the order of $1/\varepsilon$.

For further analysis, we could write a Fokker-Planck-Kolmogorov equation for the transition probability for the average process on the basis of system (5) of stochastic equations: in some cases, however, we can find the unknown quantities directly because of the nature of the average equations. As an example, we consider the system

$$\ddot{y}+(2\varepsilon\lambda(x)+\sqrt{\varepsilon}\,\dot{\xi})\,\dot{y}+k^2\,\mathrm{sgn}\,y=0,$$

$$\dot{x}=\varepsilon. \tag{7}$$

Here $\dot{\xi}$ is "white-noise" process, k = const. Equation (7) can be interpreted as a system of stochastic Ito equations. Setting $y=q,\ \dot{y}=p,$ we find

$$dq=pdt,$$

$$dp=[-k^2\,\mathrm{sgn}\,q-2\varepsilon\lambda(x)\,p]\,dt-\sqrt{\varepsilon}\,pdW,$$

$$dx=\varepsilon dt.$$

Using Eqs. (5) and (6), we find the following expression for the logarithm of F (it is assumed here that $F=\overline{F}_1=-F_2$.) :

$$\ln\frac{\overline{F}(\tau)}{F(0)}=-\frac{1}{3}\left[4\int_0^\tau\lambda(s)\,ds+\frac{\tau}{5}\right]+\frac{2}{\sqrt{5}}\,[W(\tau)-W(0)].$$

Here $\tau=\varepsilon t$, F(0) is the value of F at t = 0 (governed by the initial conditions), and $W(\tau)-W(0)$ is the increment in the Wiener process.

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