STABILITY OF ROUND PIPE FLOWS

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Vestnik Moskovskogo Universiteta. Fizika,
Vol. 30, No. 6, pp. 650-653, 1975

UDC 532.1

It is shown that stability of rotationally symmetrical perturbations of a flow of an ideal fluid in a round pipe implies that the velocities of asymmetrical perturbations are bounded.

It is known that Squire's theorem that stability of two-dimensional perturbations guarantees stability of three-dimensional perturbations is valid for a plane-parallel flow of a viscous fluid. For plane-parallel flows of an ideal fluid, Squire's theorem holds only in a greatly weakened form: stability of two-dimensional perturbations implies only bounds on the velocities of three-dimensional perturbations, and the vorticity increases with time [1,2].

If we go from plane-parallel flows to more general flows, for example, a flow in a round pipe or between cylinders, Squire's theorem fails altogether for the viscous fluid. Thus, it is known that a Couette flow between cylinders loses stability precisely in the third dimension, while two-dimensional perturbations are always stable. For a round pipe, there is good reason to believe that rotationally symmetrical perturbations (the analog of the two-dimensional perturbations in this case) are always stable, but the efforts of investigators with respect to asymmetrical perturbations have been directed to the construction of examples of instability [3]. But if we go from the viscous to the ideal fluid, the weakened version of Squire's theorem that obtained for plane-parallel flows is also preserved for the round pipe. If stability of symmetrical perturbations is proven, the velocities of unsymmetrical perturbations are also limited. This is the gist of the present paper. The method will be fully analogous to that developed in [2].

We write the linearized equations of motion in the cylindrical coordinates r, θ, z, directing the Oz axis along the axis of the pipe. We shall use u for the radial velocity component, v for the component in the direction of increasing polar angle θ, and w for the component on Oz. We have

$$\begin{gathered}\frac{\partial u}{\partial t}+W\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r},\\ \frac{\partial v}{\partial t}+W\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{r\partial\theta},\\ \frac{\partial w}{\partial t}+W\frac{\partial w}{\partial z}+uW'=-\frac{1}{\rho}\frac{\partial p}{\partial z},\\ \frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial v}{r\partial\theta}+\frac{\partial w}{\partial z}=0,\\ W'=\frac{dW}{dr}.\end{gathered}\tag{1}$$

Here W(r) is the velocity of the main stream at the boundary under the condition $u(r_0) = 0$. We shall consider perturbations that depend harmonically on the coordinates θ, z as $\exp[i(n\theta+kz)]$. We have for the amplitude functions

$$\begin{gathered}u_t+ikWu=-p_r/\rho,\\ v_t+ikWv=-inp/r\rho,\\ w_t+ikWw+W'u=-ikp/\rho,\\ (ru)_r/r+inv/r+ikw=0.\end{gathered}\tag{2}$$

We convert to equations for the amplitude functions of the vorticity components:

$$\rho_1 = inw/r - ikv, \quad \rho_2 = iku - w_r. \tag{3}$$

Differentiating these expressions with respect to t and applying (2), we get

$$\begin{gathered}
\left(\frac{\partial}{\partial t} + i_k W\right)\rho_1 + \frac{in}{r} W'u = 0, \\
\left(\frac{\partial}{\partial t} + ikW\right)\rho_2 + \frac{in}{r} W'v - r\left(\frac{W'}{r}\right)' u = 0, \\
\left(\frac{\partial}{\partial t} + ikW\right)\rho_3 + ikW'v = 0.
\end{gathered} \tag{4}$$

A certain linear combination of the vorticity components is expressed in terms of u:

$$\begin{gathered}
ikr\rho_2 - in\rho_3 - \frac{2nkr}{n^2 + k^2r^2}\rho_1 = (ru)_{rr} + \\
+ \frac{n^2 - k^2r^2}{n^2 + k^2r^2}\,\frac{(ru)_r}{r} - \frac{1}{r}(n^2 + k^2r^2)\,u.
\end{gathered} \tag{5}$$

We choose the corresponding linear combination of Eqs. (4):

$$\begin{gathered}
\left(\frac{\partial}{\partial t} + ikW\right)\left[(ru)_{rr} + \frac{n^2 - k^2r^2}{n^2 + k^2r^2}\,\frac{(ru)_r}{r} - \frac{1}{r}(n^2 + k^2r^2)\,u\right] - \\
- ik\left[r^2\left(\frac{W'}{r}\right)' + \frac{2n^2}{n^2 + k^2r^2} W'\right] u = 0.
\end{gathered} \tag{6}$$

It is convenient to endow the operator on r that appears in the first brackets with self-adjoint form by substitution of variables. For this purpose we put

$$u = a(r)\,\xi, \quad a(r) = \sqrt{(n^2 + k^2r^2)/r^3}. \tag{7}$$

We obtain an equation for ξ:

$$\left(\frac{\partial}{\partial t} + ikW\right)(\xi_{rr} - \varphi(r)\,\xi) - ikc\xi = 0, \tag{8}$$

where

$$\begin{gathered}
\varphi(r) = \frac{1}{4(n^2 + k^2r^2)^2}\left[\frac{n^4}{r^2}(4n^2 - 1) + 2n^2k^2(6n^2 - 5) + \right. \\
\left. + 3r^2k^4(4n^2 + 1) + 4r^4k^6\right]
\end{gathered} \tag{9}$$

and

$$c(r) = r\left(\frac{W'}{r}\right)' + \frac{2n^2W'}{r(n^2 + k^2r^2)} = W'' + \frac{n^2 - k^2r^2}{r(n^2 + k^2r)} W'. \tag{10}$$

It is obvious that $\varphi(r)$ is positive for all n and k. As for c(r), we shall henceforth assume that it is nonvanishing. This condition is fully equivalent to Rayleigh's condition of the absence of inflection points on the velocity profile in the plane-parallel case, which guarantees stability of two-dimensional perturbations. For the parabolic profile of greatest interest

$$W = W_0(r_0^2 - r^2)$$

we have

$$c(r) = 4W_0n^2/(n^2 + k^2r^2). \tag{11}$$

At n = 0 (rotationally symmetrical perturbations), $c(r) \equiv 0$. For the other n we have c(r) < < 0. For symmetrical perturbations, the situation is the same as for the plane-parallel Couette flow: Eq. (8) degenerates

$$\left(\frac{\partial}{\partial t} + ikW\right)(\xi_{rr} - \varphi\xi) = 0, \tag{12}$$

from which

$$\xi_{rr} - \varphi(r)\,\xi = f(r)\,e^{-ikW(r)t}.$$

Thus, $\xi_{rr}-\varphi(r)\xi$ is bounded for all t and is uniform with respect to r. It is easily seen that ξ is also bounded. For asymmetrical perturbations, when $c \neq 0$, we obtain the conservation law

$$\frac{d}{dt}\int_0^{r_0}\left\{|\xi_r|^2+\varphi|\xi|^2+\frac{W-K}{c(r)}|\xi_{rr}-\varphi\xi|^2\right\}dr=0, \tag{13}$$

where K is an arbitrary constant (this law can be verified directly). The integral in (13) is a function only of the single variable t and d/dt is the ordinary derivative of a function of one variable. The constant K can be so chosen that $(W-K)/c>0$. Hence follows boundedness in the rms values $\xi/\sqrt{\varphi}$, ξ_r and $\xi_{rr}-\varphi\xi$. We obtain from the first equation of (4) and from (8)

$$\left(\frac{\partial}{\partial t}+ikW\right)\left[kc\rho_1+\frac{naW'}{r}(\xi_{rr}-\varphi\xi)\right]=0,$$

from which

$$\rho_1=\frac{naW'}{kcr}(\xi_{rr}-\varphi\xi)+f(r)e^{ikWt}. \tag{14}$$

Therefore ρ_1 is bounded in the root-mean-sqare by the same constant for all t (it is assumed at all times that the initial perturbations are bounded in the same sense). Now from $u\sim r^{-1/2}\xi$

$$v=\frac{i}{k^2r^2+n}[n(ru)_r-k^2\rho_1]\sim[kr(r^{-1/2}\xi)_r-nr\rho_1] \tag{15}$$

and boundedness of $\xi/\sqrt{\varphi}\sim\xi/r$ and ξ_r in the rms implies boundedness of the integrals

$$\int_0^{r_0}|u|^2rdr,\quad \int_0^{r_0}|v|^2rdr,\quad \int_0^{r_0}|w|^2rdr \tag{16}$$

by the same constant, which is independent of t (if these integrals existed at t = 0), Q.E.D. The linear increase of vorticity is established in the same way as in [2]. It follows from the fact that although the quantities of the form $f(r)e^{-ikWt}$ are limited, their derivatives with respect to r increase linearly with t.

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30 January 1974

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