THEORY OF MULTIPLE SCATTERING IN A SEMIINFINITE SPATIALLY-INHOMOGENEOUS MEDIUM (II)

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Two simple models of the spatial dependence Λ(z) are used with first- and second-order perturbation theory (developed in [1]) to determine the intensity of the emerging radiation and the coefficient of reflection in the case of monochromatic scattering in a semiinfinite medium. Higher-order perturbation theory is briefly discussed. The results obtained can be used in model calculations to take into account the roughness of the surface of a light-scattering medium, and to determine the light-scattering characteristics of diffusely reflecting dyes as functions of surface properties.

A perturbation theory involving the number of scatters in an "inhomogeneous region" was developed in [1]. In this paper, we consider the application of the results of that paper to the determination of the intensity of emerging radiation and the reflection coefficient of a semiinfinite ($0 \leq z < \infty$), monochromatically scattering medium whose reflection and extinction coefficients are rapidly varying functions of z in an "inhomogeneity." The inhomogeneity is assumed to be localized near the z = 0 surface and its shape is described by two simple models introduced below. The validity of any particular model must, of course, depend on the nature of the inhomogeneity, and the model must be chosen on the basis of a priori (usually nonoptical) considerations. The formulas obtained below can be used to isolate explicitly the effect of the inhomogeneity in first- and second-order perturbation theory, i.e., they enable us to solve the converse scattering problem and to determine the optical characteristics of the medium in the region of the inhomogeneity in terms of the incident and emerging intensities. Multiple scattering of light outside the inhomogeneity is isolated and rigorously taken into account.

FIRST-ORDER PERTURBATION THEORY

A perturbation theory series was constructed in [1] for the source function $f(\tau, \mu_0)$, i.e., the intensity $J(z(\tau), \mu, \mu_0)$ at optical depth τ from the surface, averaged over all angles arccos μ_0. A formula was also obtained ([1], (15)) for the intensity of the emerging radiation $J(0, \mu, \mu_0) \equiv J(\mu, \mu_0)$ (in the presence of the perturbation) in the form of a single integral containing the source function in a linear form. Since the n-th term of the series for the source function ([1], (29)) is expressed in the form of an n-fold integral, the corresponding term of the series for the emerging intensity has the form of an (n + 1)-fold integral. Using the properties of the resolvent, we can calculate the outer integral and reduce the order of integration by unity. In general, this is done in precisely the same way as in the first order, which we shall now consider.

In the expression ([1], (15)) for the radiation intensity $J(\mu, \mu_0)$ emerging at an angle arccos μ to the normal to the surface of the medium, we isolate the perturbing term which is proportional to $Q(\tau)$ ([1], (24)) and obtain

$$J(\mu, \mu_0) = \frac{\Lambda}{2\mu}\left\{\int_0^\infty e^{-\tau/\mu} f(\tau, \mu_0)\, d\tau + \int_0^\infty Q(\tau) e^{-\tau/\mu} f(\tau, \mu_0)\, d\tau\right\}. \quad (1)$$

We recall that μ_0 represents the cos of the angle of incidence of the radiation to the inward normal to the surface of the scattering medium. Substituting the expansion given by

([1], (29)) by (1), and retaining only first-order terms, we obtain

$$J^{(0)}(\mu, \mu_0) - J(\mu, \mu_0) = \frac{\Lambda}{2\mu}\int_0^\infty Q(\tau) f^{(0)}(\tau, \mu_0) \left\{ e^{-\tau/\mu} + \int_0^\infty e^{-\tau'/\mu} \Gamma(\tau'|\tau) d\tau' \right\} d\tau, \quad (2)$$

where $J^{(0)}$ is the undisturbed intensity.

Since the resolvent $\Gamma(\tau|\tau')$ is symmetric, it is readily seen that the expression in braces in (2) is the unperturbed source function $f^{(0)}(\tau, \mu)$ corresponding to radiation of intensity $I_0=2\pi$ incident at the angle arccos μ (and not μ_0, as usual). Hence we obtain the following formula which is very symmetric in μ, μ_0:

$$J^{(0)}(\mu, \mu_0) - J(\mu, \mu_0) = \frac{\pi\Lambda}{I_0\mu}\int_0^\infty f^{(0)}(\tau, \mu_0) Q(\tau) f^{(0)}(\tau, \mu) d\tau. \quad (3)$$

It follows that whatever the particular form of $Q(\tau)$, the first-order correction to the emerging radiation (multiplied by μ, which is connected with the normalization of the incident radiation) is a symmetric function of μ, μ_0.

Let us now consider the case where the inhomogeneity is localized in a narrow, deep ($h \ll 1$) region on the surface at $z = 0$. Since $h \ll 1$ and the perturbation $Q(\tau) = -K(\tau)/(1 + K(\tau))$ is appreciably nonzero only for $\tau \leqslant h$, the integral in (3) can be evaluated by using the asymptotic form of the source functions $f^{(0)}(\tau, \mu)$ and $f^{(0)}(\tau, \mu_0)$ for $\tau \to 0$. To determine this asymptotic behavior, we use the expression for $f^{(0)}$ obtained with the aid of the standard source function $\Phi(\tau, \Lambda) \equiv \Gamma(\tau|0)$.

According to [2, 3], we have

$$f^{(0)}(\tau, \mu_0) = \frac{I_0}{2\pi}\varphi(\mu_0, \Lambda)\left\{ e^{-\tau/\mu_0} + \int_0^\tau e^{-\frac{\tau-\tau'}{\mu_0}} \Phi(\tau', \Lambda) d\tau' \right\}, \quad (4)$$

where $\varphi(\mu_0, \Lambda)$ is the well-known Ambartsumyan function [2-5]. Since $\Phi = O(\ln \tau)$ for $\tau \to 0$, the second term in (4) is $O(h \ln h)$ when $\tau \leqslant h$. Neglecting this term in comparison with unity, we obtain

$$f^{(0)}(\tau, \mu_0) \backsim \frac{I_0}{2\pi}\varphi(\mu_0, \Lambda) e^{-\tau/\mu_0}. \quad (5)$$

If we now let $\Delta J = J - J^{(0)}$ and substitute (5) in (3), we obtain

$$\Delta J = -\frac{I_0}{4\pi\mu}\varphi(\mu_0)\varphi(\mu)\int_0^\infty Q(\tau) e^{-\tau(1/\mu+1/\mu_0)} d\tau. \quad (6)$$

The integral in (6) is the Laplace transform of the perturbation $Q(\tau)$ and can be explicitly evaluated for a broad class of functions $Q(\tau)$. However, we shall confine our attention to the two most interesting sample models, namely, the narrow step functions

$$K_1(\tau) = \begin{cases} b, & 0 \leqslant \tau \leqslant h; \\ 0, & \tau > h \end{cases} \qquad Q_1(\tau) = -\begin{cases} \dfrac{b}{1+b}, & 0 \leqslant \tau \leqslant h \\ 0, & \tau > h \end{cases} \quad (7)$$

and the exponentially decaying functions

$$K_2(\tau) = a/(e^{-\tau/h} - a); \quad Q_2(\tau) = -ae^{-\tau/h}; \quad (a < 1). \quad (8)$$

Both models are very satisfactory in the case of light-scattering dyes with time-dependent properties near the surface.

It is well known [2-5] that the unperturbed intensity $J^{(0)}(\mu, \mu_0)$ can be expressed in terms of the Ambartsumyan functions as follows:

$$J^{(0)}(\mu,\ \mu_0) = \frac{I_0\Lambda}{4\pi}\,\frac{\varphi(\mu)\,\varphi(\mu_0)}{\mu+\mu_0}\,\mu_0. \tag{9}$$

Hence, we obtain from (6) and (7) the following expression for $\Delta J \equiv \Delta J_1$ in the case of the step form of $Q(\tau)$:

$$\Delta J_1(\mu,\ \mu_0) = -J^{(0)}(\mu,\ \mu_0)\,\frac{b}{1+b}\left[1-e^{-h(1/\mu+1/\mu_0)}\right]. \tag{10}$$

For the exponential model, we have similarly

$$\Delta J_2(\mu,\ \mu_0) = -J^{(0)}(\mu,\ \mu_0)\,\frac{a\cdot(\mu+\mu_0)\cdot h}{\mu\mu_0+h\cdot(\mu+\mu_0)}. \tag{11}$$

If we expand the exponential in (10) and retain only the leading term in h in (10) and (11), we find that ΔJ_1 and ΔJ_2 are equal when the "heights" of the step functions $K_1(\tau)$ and $K_2(\tau)$ are equal, i.e., $a = b/(1 + b)$.

In general, it follows from (10) and (11) that $\Delta J = O(h)$ when $\mu = O(1)$ and $\mu_0 = O(1)$. Conversely, in the region where $\mu_0 = O(h)$ or $\mu = O(h)$, we have $\Delta J = O(1)$, i.e., the corrections are large when the emerging or incident rays are nearly tangent to the surface $\tau = 0$. The same result can be obtained from (6) for functions $Q(\tau)$ other than those given by (7) or (8).

Equation (10) has the following simple interpretation. The lifetime of a photon entering a thin layer of thickness h at an angle μ_0 is e^{-h/μ_0}, and the lifetime of a photon emerging at an angle arccos μ is $e^{-h/\mu}$. It follows that the expression in brackets in (10) gives the probability of absorption in this layer. Next, since in the first order we take into account only single scatters in the layer $0 \le \tau \le h$, we have two alternatives: either the photon interacts with a scattering center in the layer, or it interacts with an absorbing center. The factor $b/(1 + b)$ takes into account the corresponding probability. In other words, this factor gives the probability that the photon will continue to move in a straight line in the layer. If, however, the photon has already been scattered, then in the first order it will not interact again in the given layer. For the step model, the formula given by (10) could have been derived from the elementary probabilist considerations introduced above, but this would have been difficult for the exponential model (8).

There is considerable interest in the change $\Delta R^{(1)}(\mu_0,\ \Lambda)$ in the reflection coefficient due to the appearance of an inhomogeneity:

$$\Delta R^{(1)}(\mu_0,\ \Lambda) = \frac{2\pi}{I_0}\int_0^1 \Delta J(\mu,\ \mu_0)\,\mu\,d\mu. \tag{12}$$

In the case of an exponential inhomogeneity of the form given by (8), we substitute $h_1 = h/(\mu_0 + h)$ so that we have from (11) and (12)

$$\Delta R^{(1)}(\mu_0,\ \Lambda) = -\frac{\Lambda a h_1}{2}\,\varphi(\mu_0)\,\mu_0\left\{\int_0^1 \varphi(\mu)\,d\mu - h_1\int_0^1 \frac{\varphi(\mu)\,d\mu}{\mu+h_1}\right\}. \tag{13}$$

The first term in the braces represents the zero-order moment $\alpha_0(\Lambda)$ of the function $\varphi(\mu,\Lambda)$, which is known to have the form [2]

$$\alpha_0(\Lambda) \equiv \int_0^1 \varphi(\mu)\,d\mu = \frac{2}{\Lambda}\left(1-\sqrt{1-\Lambda}\right).$$

The second term is exactly the same as the integral term in the Ambartsumyan equation [2], and if we use it we finally obtain

$$\Delta R^{(1)}(\mu_0,\ \Lambda) = -a h_1\cdot\varphi(\mu_0,\ \Lambda)\,\mu_0\left[1-\sqrt{1-\Lambda}-\frac{\varphi(h_1,\ \Lambda)-1}{\varphi(h_1,\ \Lambda)}\right]. \tag{14}$$

This equation is useful for $\mu_0 \backsim h$, i.e., $h_1 \backsim 1$ and $\varphi(h_1, \Lambda) - 1 \backsim 1$. It can be substantially simplified for $\mu_0 \gg h$:

$$\Delta R^{(1)} = -ah\varphi(\mu_0, \Lambda)(1 - \sqrt{1-\Lambda}). \tag{15}$$

When the inhomogeneity is of the form given by (7), the reflection coefficient is much more difficult to determine because the convergence of the integral with respect to μ in (12), which contains the exponential from (10), is highly nonuniform. We shall therefore use the integral representation ([1], (15)) with $K(\tau) = 0$ for $I^{(0)}(\mu, \mu_0)$. We begin by carrying out the integration with respect to μ. The result is

$$\Delta R^{(1)} = \frac{b}{1+b} R^{(0)} - \frac{2\pi}{I_0} \frac{be^{-h/\mu_0}}{1+b} \frac{\Lambda}{2} \int_0^\infty f^{(0)}(\tau, \mu_0) E_2(\tau + h)\, d\tau, \tag{16}$$

where $E_n(t) = \int_0^1 e^{-t/\mu} \mu^{n-2}\, d\mu$, as in [3] and $R^{(0)}$ is the unperturbed reflection coefficient which can be written in the analogous form

$$R^{(0)}(\mu_0, \Lambda) = \frac{2\pi}{I_0} \cdot \frac{\Lambda}{2} \int_0^\infty f^{(0)}(\tau, \mu_0) E_2(\tau)\, d\tau. \tag{17}$$

Expanding $E_2(\tau + h)$ in (16) (in powers of h) and using (17), we obtain the following expression which is valid to within first-order terms:

$$\Delta R^{(1)} = \frac{bR^{(0)}}{1+b}(1 - e^{-h/\mu_0}) + \frac{2\pi h}{I_0} \frac{be^{-h/\mu_0}}{1+b} \int_0^\infty f^{(0)}(\tau, \mu_0) \frac{dE_2(\tau)}{d\tau}\, d\tau.$$

Since $dE_2(\tau)/d\tau = -E_1(\tau)$ is the kernel of the integral equation for $f^{(0)}(\tau, \mu_0)$, the integral in the last formula can be evaluated exactly and, if we use (4) or (5), we obtain

$$\Delta R^{(1)} = \frac{R^{(0)}b}{1+b}(1 - e^{-h/\mu_0}) + \frac{hb}{1+b}[\varphi(\mu_0) - 1]. \tag{18}$$

It is interesting that both (14) and (18) split into a pair of terms that behave differently for $\mu_0 \backsim h$. The first term is O(1) and the second O(h ln h). Thus, even a small change in the optical properties of the surface layer on the semi-infinite scattering medium gives rise to a radical change in the reflection coefficient for angles of incidence near to 90°.

SECOND-ORDER PERTURBATION THEORY

When the probability of interaction between a photon and the "new" centers forming the inhomogeneity is small, we can restrict our attention to first-order perturbation theory. This condition is definitely satisfied when the inhomogeneity is localized in an exceedingly narrow surface layer, which occurs, for example, when the light-scattering medium is damaged by ultraviolet or low-energy corpuscular ionizing radiation. However, when the accuracy of first-order formulas has to be estimated, or when the probability of double scattering in the layer is comparable with unity, it is essential to have an expression for $\Delta J^{(2)}(\mu, \mu_0)$, i.e., the second-order correction to the intensity of emerging radiation, which is valid for all μ, μ_0. We shall find the correction $\Delta J^{(2)}(\mu, \mu_0)$ for the exponential model (8) to within leading terms in h.

When $J^{(1)}$ was calculated, we used in (6) the asymptotic form of $f^{(0)}(\tau, \mu)$ and $f^{(0)}(\tau, \mu_0)$ for $\tau \to 0$, i.e., we neglected the second term in (4). Let us denote by $\Delta^{(12)}J(\mu, \mu_0)$ the error in the first-order correction, which is introduced by this procedure. Since $\Delta^{(12)}J(\mu, \mu_0) = O(h^2 \ln h)$ is of the same order as the leading terms (in h) of the second-order perturbation-theory correction $\Delta^{(2)}J(\mu, \mu_0)$, we must write $\Delta J^{(2)}(\mu, \mu_0)$ in the form

$$\Delta J^{(2)}(\mu, \mu_0) = \Delta^{(2)} J(\mu, \mu_0) + \Delta^{(12)} J(\mu, \mu_0). \tag{19}$$

According to ([1], (29)) and ([1], (15)), we have the following expression for $\Delta^{(2)}J(\mu, \mu_0)$:

$$\Delta^{(2)} J(\mu, \mu_0) = \frac{\Lambda}{2\mu} \left\{ \int_0^\infty e^{-\tau/\mu} f^{(2)}(\tau, \mu_0)\, d\tau + \int_0^\infty Q(\tau) e^{-\tau/\mu} f^{(1)}(\tau, \mu_0)\, d\tau \right\}. \tag{20}$$

The quantities $f^{(1)}(\tau, \mu_0)$ and $f^{(2)}(\tau, \mu_0)$ are the corresponding terms of the iteration series ([1], (29)). Proceeding as between (2) and (3), and substituting $J_0 = 2\pi$ for the sake of brevity, we obtain

$$\frac{2\mu}{\Lambda} \Delta^{(2)} J = a^2 \int_0^\infty\int [f^{(0)}(\tau_0, \mu_0) - e^{-\tau_0/\mu_0}]\, e^{-\frac{\tau_0+\tau_1}{h}} f^{(0)}(\tau_1, \mu)\, \Gamma(\tau_0 | \tau_1)\, d\tau_0 d\tau_1 +$$
$$+ a^2 \int_0^\infty e^{-\tau_0/h} f^{(0)}(\tau_0, \mu_0) [f^{(0)}(\tau_0, h^{(1)}(\mu)) - e^{-\tau_0/h^{(1)}(\mu)}]\, d\tau_0,$$

where

$$h^{(1)}(\mu) = h\mu/(h + \mu).$$

We now retain only the leading terms in h, use the properties of the resolvent and the well-known equation [2, 3]

$$\int_0^\infty e^{-\tau/x} f^{(0)}(\tau, x')\, d\tau = \frac{\varphi(x)\varphi(x')}{x + x'} xx', \tag{21}$$

and substitute

$$h(\mu, \mu_0) = h^{(1)}(\mu) h^{(1)}(\mu_0)/[h^{(1)}(\mu) + h^{(1)}(\mu_0)],$$

we finally obtain

$$\Delta^{(2)} J = \frac{\Lambda a^2}{2\mu} [\varphi(\mu_0) - \varphi(\mu) + \varphi(\mu_0)\varphi(\mu)] [\varphi(h^{(1)}(\mu))\, \varphi(h^{(1)}(\mu_0)) - 1]\, h(\mu, \mu_0). \tag{22}$$

It is interesting to note that when $\mu \ll h$, the correction $\Delta^{(2)}J$ is

$$\Delta^{(2)} J(0, \mu_0) = \frac{\Lambda a^2}{2} [2\varphi(\mu_0) - 1] [\varphi(h^{(1)}(\mu_0)) - 1] = O(h \ln h).$$

Thus, for rays that are almost tangential to the surface, there is a sharp increase in the correction in both the first- and second-order even though $\Delta^{(2)}J$ is $O(h \ln h \cdot \Delta J^{(1)})$.

To complete the calculation of $\Delta J^{(2)}$, we must find the quantity $\Delta^{(12)}J(\mu, \mu_0)$ which can readily be reduced to the form

$$\Delta^{(12)} J = -\frac{\Lambda a}{2\mu} \varphi(\mu_0) \int_0^\infty \Phi(\tau', \Lambda)\, e^{\tau'/\mu_0} d\tau' \int_0^\infty e^{-\tau/h^{(1)}(\mu_0)} f^{(0)}(\tau, \mu)\, d\tau. \tag{23}$$

Since $f^{(0)}(\tau, \mu)$ varies slowly in comparison with $e^{-\tau/h}$, it can be taken outside the inner integral. We thus obtain

$$\Delta^{(12)} J = -\frac{\Lambda a}{2\mu} \varphi(\mu_0) h^{(1)}(\mu_0) \int_0^\infty \Phi(\tau', \Lambda) e^{-\tau'/h} f^{(0)}(\tau', \mu) d\tau'.$$

Since $\Phi(\tau|, \Lambda) = \Gamma(\tau|0)$, we can proceed in the usual way to show that

$$\Delta^{(12)} J(\mu, \mu_0) = \frac{\Lambda a}{2\mu} \varphi(\mu) \varphi(\mu_0) [\varphi(h^{(1)}(\mu)) - 1] h^{(1)}(\mu_0). \quad (24)$$

It follows from (23) and (24) that $\Delta^{(12)}J = O(h^2 \ln h)$ or $\mu \geq h$, and when $\mu < h$, we have $\Delta^{(12)}J(\mu, \mu_0) = O(h \ln h)$. Hence it follows, in particular, that the former region provides a contribution $O(h^2 \ln h)$ to the reflection coefficient whereas the latter region, i.e., the region of small μ, provides a contribution $O(h^3 \ln h)$ due to the integration over $\mu d\mu$. Hence the singularity of $\Delta^{(2)}J$ and $\Delta^{(12)}J$ for small μ is unimportant for the second-order corrections to the reflection coefficient, which we shall now consider.

According to (19), (22), and (24), the quantity $\Delta R^{(2)}$ splits into two terms, the first of which is proportional to a^2 and the second to a:

$$\Delta R^{(2)}(\mu_0, \Lambda) = \Delta^{(2)} R(\mu_0, \Lambda) + \Delta^{(12)} R(\mu_0, \Lambda). \quad (25)$$

We now evaluate (12) to within $O(h^2)$ and substitute $h^{(1)}(\mu) \approx h$ in (22) and (24). This yields

$$\Delta^{(2)} R(\mu_0, \Lambda) = \frac{a^2 h h^{(1)}(\mu_0)}{h + h^{(1)}(\mu_0)} \cdot \frac{\Lambda}{2} [\varphi(h) \varphi(h^{(1)}(\mu_0)) - 1] [\varphi(\mu_0) + (\varphi(\mu_0) - 1) \alpha_0(\Lambda)[\quad (26)$$

and

$$\Delta^{(12)} R(\mu_0, \Lambda) = \frac{a\Lambda}{2} [\varphi(h) - 1] \varphi(\mu_0) h^{(1)}(\mu_0) \alpha_0(\Lambda). \quad (27)$$

When $\mu_0 \sim 1$, Eqs. (26) and (27) are simplified and the total correction to the reflection coefficient assumes the form

$$\Delta R^{(2)} = \frac{\Lambda a h}{2} [\varphi(h) - 1] \left\{ a \frac{[\varphi(h) + 1}{2} [\varphi(\mu_0)(1 + \alpha_0) - \alpha_0] + \varphi(\mu_0) \alpha_0 \right\}. \quad (28)$$

Of course, when (25)-(28) are used to determine the second-order correction, the first-order correction must be calculated from (14) and not from (15).

HIGHER-ORDER PERTURBATION THEORY

To investigate the dependence of $J(\mu, \mu_0)$ on the concentration of the "new" centers in the inhomogeneity, i.e., in the final analysis, the dependence of a, it may be useful to have higher-order perturbation-theory corrections. Using the above technique, let us consider the entire perturbation-theory series, retaining only the leading terms in h in the coefficients of powers of a. Let $\Delta f(\tau) \equiv f(\tau, \mu_0) - f^{(0)}(\tau, \mu_0)$ and $I_0 = 2\pi$, so that, subject to the above assumptions, the perturbation-theory series ([1], (29)) for $Q(\tau)$ in the form given by (8) takes the form

$$\Delta f(\tau) = \sum_{n=1}^{\infty} (-1)^n a^n [f^{(0)}(\tau, h^{(n-1)}) - e^{-\tau/h^{(n-1)}}] \prod_{m=1}^{n} [\varphi(h^{(m-2)}) - 1], \quad (29)$$

where for n = 0, 1, 2, ...

$$h^{(n)}(\mu_0) = \frac{h\mu_0}{h + \mu_0 n}; \quad h^{(n)} \sim \frac{h}{n} \quad \text{при} \quad \mu_0 = 1 \tag{30}$$

and we have made the formal substitution $\varphi(h^{(-1)}) \equiv 2$.

The formula given by (29) reveals two interesting features of the perturbation-theory series ([1], (29)) which one might expect on the basis of physical considerations (or, more precisely, from the fact that, in ([1], (29)), scattering outside the inhomogeneity was taken into account exactly). As the order n increases, we gradually lose all the information on the initial angle of incidence arccos μ_0 and each successive term of the series describes the variation in $f(\tau, \mu)$ for smaller values of τ because h goes over into h/n. In other words, we may say that, for any given μ_0, the inhomogeneity region becomes negligibly narrow.

Let us now consider the intensity of emerging radiation which we shall write in the form $\Delta J = J - J^{(0)}$. We have

$$\Delta J = -a\int_0^\infty e^{-\tau/h^{(1)}(\mu)} f^{(0)}(\tau, \mu_0)\,d\tau + \int_0^\infty e^{-\tau/\mu}(1 - ae^{-\tau/h})\,\Delta f(\tau)\,d\tau. \tag{31}$$

The first integral can be evaluated in an elementary fashion in accordance with (21), and if we denote the second integral in (31) by ΔJ_0, we have $\Delta J_0 = \Delta J_1 - a\Delta J_2$ where ΔJ_2 is obtained from ΔJ_1 by substituting $\mu \to h^{(1)}(\mu)$. If we suppose that $A_n(\mu_0)$ is equal to the product in (29), we can write ΔJ_1 in the form

$$\Delta J_1 = \sum_{n=1}^\infty (-a)^n A_n(\mu_0) \int_0^\infty [f^{(0)}(\tau, h^{(n-1)}) - e^{-\tau/h^{(n-1)}}]\,e^{-\tau/\mu}\,d\tau. \tag{32}$$

Hence, using (21), we obtain

$$\Delta J_1 = \sum_{n=1}^\infty (-a)^n A_n(\mu_0) \frac{\mu h^{(n-1)}(\mu_0)}{\mu + h^{(n-1)}(\mu_0)} [\varphi(\mu)\,\varphi(h^{(n-1)}) - 1]. \tag{33}$$

Similarly,

$$\Delta J_2 = \sum_{n=1}^\infty (-a)^n A_n(\mu_0) \frac{h^{(1)}(\mu)\,h^{(n-1)}(\mu_0)}{h^{(1)}(\mu) + h^{(n-1)}(\mu_0)} [\varphi(h^{(1)}(\mu))\,\varphi(h^{(n-1)}(\mu_0)) - 1]. \tag{34}$$

If, finally, we use the asymptotic behavior of the Ambartsumyan function for $\mu \to 0$, i.e. $\varphi(\mu) \approx 1 - \frac{\Lambda}{2}\mu\ln\mu$ and suppose that $\mu\mu_0 \sim 1$, then we finally obtain the following expressions for ΔJ_1 and ΔJ_2:

$$\Delta J_1 = [1 - \varphi(\mu)]\left\{ha + \frac{2}{\Lambda}\sum_{n=2}^\infty \frac{(a\Lambda h)^n}{2^n n!}\prod_{m=2}^n \ln\left(\frac{h}{m-1}\right)\right\} \tag{35}$$

and

$$\Delta J_2 = [1 - \varphi(h^{(1)}(\mu))]\left\{ha + \frac{2}{\Lambda}\sum_{n=2}^\infty \frac{(a\Lambda h)^n \cdot n}{2^n (n+1)!}\prod_{m=2}^n \ln\left(\frac{h}{m-1}\right)\right\}. \tag{36}$$

These two formulas are sufficiently simple for approximate summation in the case of particular problems.

We note in conclusion that the formalism developed in this and in the previous paper can be used in model calculations of the roughness of the surface of a light-scattering medium if it can be represented by a thin layer with optical parameters differing from the parameters in the interior of the medium. The general conclusion that this type of "inhomogeneity" has a dominating effect on the intensity for small μ, μ_0 in all orders of perturbation theory remains valid.

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