NEW SOLUTIONS OF THE SOLIDIFICATION AND MELTING PROBLEMS

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A method is developed for the solution of the one-dimensional problem of thermal conduction with the moving boundary conditions, namely, the problem of solidification and melting. By transforming the variables, the problem is reduced to that of fixed boundary conditions. Examples of solutions are reproduced.

The cooling of a molten material, followed by solidification, and the heating of a solid material followed by melting are two problems in thermal physics that are being intensively investigated at the present time. Both problems are of considerable interest to heavy industry. The English metallurgist Lightfoot [1] was one of the first to consider the solidification problem in 1930.

The characteristic feature of these problems is the fact that the boundary conditions are formulated on moving surfaces. In this paper, we give a solution based on transformations that reduce the problem with moving boundary conditions to the problem with fixed boundary conditions.

The solution of the first problem is the simpler. Let us write down the equation for one-dimensional cooling prior to solidification in the form

$$T = T_m \cdot \Phi\left(\frac{x}{2a\sqrt{t}}\right), \tag{1}$$

where

$$a^2 = \frac{\Lambda}{c_p \rho}, \quad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z'^2} dz',$$

Λ is the thermal conductivity, c_p is the specific heat, and ρ is the density of the material. The function given by (1) satisfies the thermal conduction equation

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}$$

([2], sec. 18, pp. 42-46), and also the initial and boundary conditions for cooling: $T = T_m$ for $t = 0$ and $x \neq 0$, and $T = 0$ for $x = 0$.

The solution is valid prior to the onset of solidification. New methods have to be used after the solidification process begins (see [2, 3] and also [4]). Calculations carried out by the present author are reproduced below.

The solution of the problem of one-dimensional heating prior to the onset of melting can be written in the form

$$T = T_f \cdot \Phi^*\left(\frac{x}{2a\sqrt{t}}\right),$$

where

$$\Phi^*(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-z'^2} dz', \quad \Phi^*(z)$$

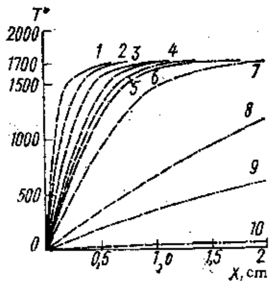

Fig. 1. Cooling of solidified medium T_m = 1700°C, heatsink at T_0 = 0°C: 1) 0.1; 2) 0.2; 3) 0.4; 4) 0.6; 5) 0.8; 6) 1.0; 7) 2.0; 8) 16; 9) 60; 10) 900.

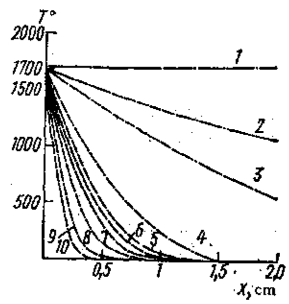

Fig. 2. Heating of solid medium by furnace: T_f = 1700°C. Notation as in Fig. 1.

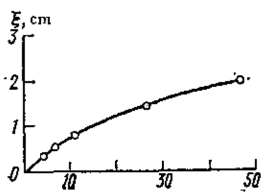

Fig. 3. Melting and solidification of steel components as functions of time.

is the probability integral of the second kind, which can be expressed in terms of the probability integral of the first kind $\Phi^*(z) \equiv 1 - \Phi(z)$.

The solution satisfies the thermal conduction equation [4, 5], and also the initial and boundary conditions $t = 0$, $x \neq 0$, $T = 0$ and $x = 0$, $T = T_f$.

The solution is valid prior to the onset of melting. Once melting begins, the problem has to be tackled by the methods considered in [2] or [3].

Figures 1 and 2 show the temperature distributions $T = T_m \cdot \Phi(z)$ and $T = T_f \cdot \Phi^*(z)$ and also the dependence of the variable x on the time t and the parameter z.

The main difficulty, namely, the time dependence of the boundary conditions, which is associated with the motion of the solidification and melting fronts, has been overcome by an analytic device consisting of moving the solidifying or melting material in the direction opposite to that of the motion of the fronts. This transformation ensures that the solidification or melting front becomes conditionally stationary. It is important to note that this operation is valid only for a semiinfinite quantity. The motion of the body gives rise to the motion of the infinitely distant boundary but the thermal effect of this boundary has no appreciable influence on the solidification or melting process. The displacement of the distant boundary can therefore be neglected.

Landau [3] has used a more general transformation of the variables in connection with the melting of a wall of finite thickness [2]. Sanders [6] used the results given in [3] in calculations involving short rods.

The transformation of the boundary condition modifies the form of the heat transfer equation. The thermal conduction equation takes the form of the convection equation

$$\frac{\partial T}{\partial t} - \dot{\xi} \frac{\partial T}{\partial x} = a^2 \frac{\partial^2 T}{\partial x^2}, \qquad (2)$$

where $\dot{\xi}$ is the velocity of the front and $a^2 = \Lambda / c_p \rho$.

Equation (2) is readily integrated. Its solution can be obtained from the solution of the thermal conduction equation by applying a similar transformation to the boundary condition.

For the unmelted material $x \geq \xi$, the solution of (2) takes the form

$$T = T_m \cdot \Phi^*\left(\frac{x - \xi}{2a\sqrt{t}}\right), \qquad (3)$$

where $\Phi^*(z)$ is the probability integral of the second kind:

$$\Phi^*(z) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z'^2} dz.$$

For the molten material $x \geq \xi$, Eq. (2) can be used but with other values of the physical parameters

$$a_1^2 \equiv \frac{\Lambda}{c_p \rho}.$$

The general solution of (2) in the present case is therefore given by the infinite sums of partial solutions:

$$T^{(1)} = T_f \sum_{k=1}^{\infty} \left\{ \Phi^* \left[\frac{2(k-1)\xi - x}{2a_1\sqrt{t}} \right] - \Phi^* \left[\frac{2k\xi + x}{2a_1\sqrt{t}} \right] \right\}. \qquad (4)$$

$$T^{(2)} = T_m \sum_{n=1}^{\infty} \left\{ \Phi^* \left[\frac{(2n-1)\xi - x}{2a_1\sqrt{t}} \right] - \Phi^* \left[\frac{(2n-1)+x}{2a_1\sqrt{t}} \right] \right\}. \qquad (5)$$

The initial temperature is assumed uniform and equal to zero. The series given by (4) and (5) are generalizations of the Carslaw theories ([5], sec. 100, pp. 231-233, Eq. (3)). These series satisfy Eq. (2) and also the boundary and initial conditions, and can therefore be looked upon as deriving the general solution of the present problem (T_f is the temperature of the furnace and T_m is the melting point).

To determine the velocity of the solidification and melting fronts, $\dot{\xi}$, we must consider the heat balance equation

$$r\rho \frac{d\xi}{dt} = \Lambda_1 \frac{\partial T^{(1)}}{\partial x}\Big|_{x=0} + \Lambda_1 \frac{\partial T^{(2)}}{\partial x}\Big|_{x=\xi}, \qquad (6)$$

where r is the latent heat of melting, ρ is the density of the material, Λ_1 is the thermal conductivity of the molten material, and $T^{(1,2)}$ is the temperature distribution in the final molten part of the material (4) and (5).

Equations (4) and (5) yield the following expressions for the heat fluxes:

$$\Lambda_1 \frac{\partial T}{\partial x}\Big|_{x=0} = \Lambda_1 \frac{2}{\sqrt{\pi}} \frac{T_f}{2a_1\sqrt{t}} \left\{ 1 + 2\sum_1^\infty e^{-\left(\frac{k\xi}{a_1\sqrt{t}}\right)^2} \right\},$$

$$\Lambda_1 \frac{\partial T^{(2)}}{\partial x}\Big|_{x=\xi} = -\Lambda_1 \frac{2}{\sqrt{\pi}} \frac{T_m}{2a_1\sqrt{t}} \left\{ 1 + 2\sum_1^\infty e^{-\left(\frac{n\xi}{a\sqrt{t}}\right)^2} \right\},$$

Table 1

Speed of Solidification and Melting Fronts

η	$1+2e^{-2\eta^2}$	$O_m \cdot \eta$	(2)—(3)	$\frac{d\eta}{(4)}$	$d\xi = 5 \cdot O_m$	$\tau = e^{(6)}$	$\xi = \eta\tau$	$\sqrt{t}$, sec½	t, sec	$\dot{\xi} = \frac{(8)}{(10)}$, cm/sec
0,25	2,556	0,717	1,849	0,135	0,387	1,472	0,368	2,16	4,66	0,788
0,30	2,395	0,86	1,535	0,1955	0,560	1,750	0,526	2,56	6,55	0,798
0,35	2,225	1,00	1,225	0,2860	0,820	2,2705	0,795	3,32	11,0	0,0722
0,40	2,055	1,145	0,910	0,440	1,260	3,525	1,41	5,15	26,6	0,053
0,42	1,9882	1,205	0,7832	0,537	1,540	4,665	1,96	6,84	46,6	0,042
0,45	1,8896	1,290	0,5996	0,752	2,155	8,628	3,88	12,62	159	0,0244
0,50	1,636	1,432	0,204	2,45	7,00	1,097	$0,548 \cdot 10^3$	$1,805 \cdot 10^3$	$2,89 \cdot 10^6$	$0,19 \cdot 10^{-3}$

where $\Lambda_1 \frac{\partial T^{(1)}}{\partial x}\Big|_{x=0}$ is the rate at which heat is supplied by the furnace to the molten part of the material and $\Lambda_1 \frac{\partial T^{(2)}}{\partial x}\Big|_{x=\xi}$ is the rate at which heat is removed from the molten material.

We now transform the heat balance equation (6) as follows:

$$r\rho \frac{d\xi}{dt} = \frac{2}{\sqrt{\pi}} \frac{\Lambda_1 (T_f - T_m)}{2a_1\sqrt{t}} \left\{1 + 2\sum_1^\infty e^{-\left(\frac{n\xi}{a_1\sqrt{t}}\right)^2}\right\}. \quad (7)$$

We thus obtain

$$\frac{r\rho a_1^2}{\Lambda_1 (T_f - T_m)} \frac{\sqrt{t}}{a_1} \frac{d\xi}{dt} = \frac{1}{\sqrt{\pi}} \left\{1 + 2\sum_1^\infty e^{-\left(\frac{n\xi}{a_1\sqrt{t}}\right)^2}\right\}. \quad (8)$$

We now transform Eq. (8):

$$O_m \frac{d\xi}{d\tau} = 1 + 2\sum_1^\infty e^{-\left(2\frac{n\xi}{\tau}\right)^2}, \quad (9)$$

where

$$O_m \equiv \frac{r\rho a_1^2}{\Lambda_1 (T_f - T_m)} \sqrt{\pi_0} \quad \tau \equiv 2a_1\sqrt{t}.$$

If we now seprate the variables in (9) with the aid of the transformation

$$\frac{\xi}{\tau} \equiv \eta, \; d\xi \equiv \tau d\eta + \eta d\tau$$

we obtain

$$O_m \tau \frac{d\eta}{d\tau} + \eta = 1 + 2\sum_1^\infty e^{-(2n\eta)^2}, \quad (10)$$

where O_m = 2.865 for steel.

Substituting

$$\frac{\tau}{d\tau} \equiv \frac{1}{d\ln\tau} \equiv \frac{1}{d\varepsilon}, \; \varepsilon \equiv \ln\tau,$$

we obtain

$$O'_m d\varepsilon \; \frac{d\eta}{1 + 2\sum_1^\infty e^{-(2n\eta)^2} - O_m\eta}, \quad (11)$$

$$O'_m \equiv \frac{1}{O_m} = 0{,}349$$

We have carried out a numerical integration using the average values of the parameters of steel in the temperature range 0-1700°C. The solution for variable values of the parameters of steel will have to be considered separately.

Figure 3 and the table show the results of integration (for steel).

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